SNARKS FROM A KÁSZONYI PERSPECTIVE: A SURVEY

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Abstract. This is a survey or exposition of a particular collection of results and open problems involving snarks — simple "cubic" (3-valent) graphs for which, for nontrivial reasons, the edges cannot be 3-colored. The results and problems here are rooted in a series of papers by László Kászonyi that were published in the early 1970s. The problems posed in this survey paper can be tackled without too much specialized mathematical preparation, and in particular seem well suited for interested undergraduate mathematics students to pursue as independent research projects. This survey paper is intended to facilitate research on these problems.

1. Introduction

The Four Color Problem was a major fascination for mathematicians ever since it was posed in the middle of the 1800s. Even after it was answered affirmatively (by Kenneth Appel and Wolfgang Haken in 1976-1977, with heavy assistance from computers) and finally became transformed into the Four Color Theorem, it has continued to be of substantial mathematical interest. In particular, there is the ongoing quest for a shorter proof (in particular, one that can be rigorously checked by a human being without the assistance of a computer); and there is the related ongoing philosophical controversy as to whether a proof that requires the use of a computer (i.e. is so complicated that it cannot be rigorously checked by a human being without the assistance of a computer) is really a proof at all. For a history of the Four Color Problem, a gentle description of the main ideas in the proof of the Four Color Theorem, and a look at this controversy over whether it is really a proof, the reader is referred to the book by Wilson [Wi]. With the Four Color Problem having been "solved", it is natural to examine related "coloring" problems that might have the kind of fascination that the Four Color Problem has had.

The Four Color Theorem has a well known equivalent formulation in terms of colorings of edges of graphs: If a graph G has finitely many vertices and edges, is "cubic" ("3-valent", i.e. each vertex is connected to exactly three edges), is planar, and satisfies certain other technical conditions (to avoid trivial technicalities), then it can be "edge-3-colored", i.e. its edges can each be assigned one of three colors in such a way that no two adjacent edges have the same color.

Now if one removes the condition that G be planar but keeps all of the other conditions, then it may be the case that G *cannot* be edge-3-colored. In a *Scientific American* article by the famous mathematics expositor Martin Gardner [Ga] in 1976, the term "snark" was coined (or rather borrowed from Lewis Carroll's tale, *The Hunting of the Snark*) for such counterexamples G for which the reason for the absence of an edge-3-coloring is in a certain technical sense "nontrivial". (More on that in Definition 2.9 in Section 2 below.)

For a long time, only very few such "nontrivial" counterexamples had been known. Then in 1975, Rufus Isaacs [Is] brought to light some infinite families of such nontrivial counterexamples, and some new methods for constructing such examples. A year later, the 1976 paper of Gardner [Ga] alluded to above, popularized that work of Isaacs and established the term "snark" for such examples. Since then, snarks have been a topic of extensive research by many mathematicians.

However, there is a particular line of open questions on snarks which has received little attention during that time. Those open questions can be tackled without too much specialized mathematical preparation. In particular, those open questions seem to be reasonably well suited for undergraduate mathematics students to pursue as independent research projects — for example in an REU (Research Experience for Undergraduates) program in Mathematics. This survey or expository paper will hopefully help anyone who becomes fascinated with snarks get started relatively quickly on research on this particular collection of open problems.

The particular collection of research problems posed in this expository paper is rooted in three obscure, relatively little known papers by László Kászonyi [Ká1, Ká2, Ká3] that were published in the early 1970s. Further work on this collection of problems was done later on by the author [Br1, Br2]; and still further work was done subsequently by Scott McKinney [McK] as part of a Mathematics REU program when he was an undergraduate mathematics major. This expository paper is intended to enable the interested reader to quickly become acquainted with the relevant material in those six papers, and to get started quickly on the collection of research problems posed here.

The organization of this paper. The collection of open problems is presented in Section 7, the final section of this paper. Sections 2 through 6 are intended as preparation for those problems, and can be summarized as follows:

Section 2 gives most of the relevant basic background material and terminology.

Section 3 gives a convenient, generously detailed, (hopefully) easy-to-digest presentation of the material in the three key papers of Kászonyi [Ká1, Ká2, Ká3] that is most directly relevant to the problems posed in Section 7. Theorem 3.3, a result of Kászonyi [Ká2, Ká3], is in essence the "backbone" of this entire survey paper. Theorem 3.5 deals with a very important classic example — the Petersen graph — in connection with Theorem 3.3.

Section 4 gives a detailed presentation of some related material from [Br1, Br2] involving pentagons (cycles with five edges) in snarks.

Section 5 is intended to get the reader "oriented" to the collection of open problems posed in Section 7. In particular, it is intended to motivate and — through Theorem 5.3 and its proof — portray in a simple context the use of certain techniques (in particular, Kochol [Ko1] "superpositions" for creating snarks) that are pertinent to research on most of the problems posed in Section 7.

Section 6 briefly explains the main results of McKinney [McK] (involving techniques similar to those in the proof of Theorem 5.3).

Much of this paper can be read quickly and somewhat superficially (without working through all of the detailed arguments). However, to be able to develop the information and techniques most important for effective work on the majority of problems posed in Section 7 (or on certain other, related ones), one needs to (1) master Section 2 (which is just background material and terminology and can be absorbed pretty quickly), (2) work through carefully, with pencil and paper, with the drawing of diagrams, the proofs of Theorems 3.3, 3.5, and 5.3, and (3) read carefully all of the rest of Section 5 as well (to become well oriented to the collection of problems in Section 7).

Depending on what particular problems one chooses to work on (either ones posed in Section 7 or other, related ones), one might want to get hold of certain other references. In [Is], [Br2, Section 2], and [McK], one can find the proofs of certain results that are stated in this survey paper without proofs. From the paper of Kochol [Ko1] itself, one can learn more about Kochol "superpositions", a key tool for some of the problems posed in Section 7. For the Four Color Problem itself (from which sprouted the study of snarks as well as much other mathematics), a fascinating, easy-to-read historical account is given in the

book by Wilson [Wi].

2. Background material

All "Remarks" and informal comments, in this section and throughout this paper, are well known, standard, trivial facts, and their proofs will usually be omitted.

Section 2 here is devoted to some background terminology. Most of it is standard, well established.

In this paper, a "graph" is always an undirected graph. It is always assumed to be nonempty (i.e. to have at least one vertex). It is always assumed (typically with explicit reminder) to have only finitely many vertices and have no loops and no multiple edges.

Two or more graphs are (pairwise) "disjoint" (from each other) if no two of them have any vertex (or edge) in common.

Within a given graph, a vertex v and an edge e are said to be "connected" (to each other) if v is an endpoint of e. Two edges are said to be "adjacent" (to each other) if they are connected to the same vertex. Two vertices are said to be "adjacent" (to each other) if they are connected to (i.e. are the endpoints of) the same edge.

Within a given graph, for a given positive integer n, a given vertex v is said to be "n-valent" if it is connected to exactly n edges. A 1-valent vertex is also said to be "univalent".

The cardinality of a given set S will be denoted "card S".

Definition 2.1. Consider the group $\mathbf{Z}_2 \times \mathbf{Z}_2$ with the operation (denoted $+$) being coordinatewise addition modulo 2. Let its elements be denoted as follows:

$$0 := (0,0), \quad a := (0,1), \quad b := (1,0), \quad c := (1,1) . \tag{2.1}$$

Thus 0 is the identity element, $a + a = b + b = c + c = 0$, also $c + b = a$, and so on.

In this paper, in all "edge-3-colorings" (Definition 2.4(b) below), the "colors" will be the nonzero elements a, b, and c of $\mathbf{Z}_2 \times \mathbf{Z}_2$ as defined in (2.1). The group structure of $\mathbf{Z}_2 \times \mathbf{Z}_2$ itself has for a long time been a handy "bookkeeping" tool in connection with edge-3-colorings (see e.g. its pervasive use in [Ko1]) and also in connection with the Four Color Theorem (see [Wi]). Its use will be illustrated in a few places below, and especially in the proof of Theorem 5.3 in Section 5.

It is tacitly understood that the use of those three "colors" a, b, and c in (2.1) is only for convenience, and that the results on edge-3-colorings given in this paper trivially carry over to the use of any other choice of three "colors".

Remark 2.2. If x, y, and z are each an element of the set $\{a, b, c\}$ (the set of non-zero elements of $\mathbf{Z}_2 \times \mathbf{Z}_2$ as defined in (2.1)), then the following two statements are equivalent:

(i) x, y, and z are distinct (that is, $\{x, y, z\} = \{a, b, c\}$);

(ii) $x + y + z = 0$ (the element $(0, 0)$ of $\mathbf{Z}_2 \times \mathbf{Z}_2$).

Definition 2.3. Suppose G is a (not necessarily connected) graph that has only finitely many vertices, no loops, and no multiple edges.

(a) The set of all edges of G will be denoted $\mathcal{E}(G)$.

(b) The set of all vertices of G will be denoted $\mathcal{V}(G)$.

(c) A "cycle" in G is of course a subgraph with (for some integer $n \geq 3$) distinct vertices $v_1, v_2, \ldots, v_n$ and edges (v_i, v_{i+1}), $i \in \{1, 2, \ldots, n - 1\}$, and (v_n, v_1). For a given integer $n \geq 3$, an "n-cycle" is a cycle with exactly n vertices. A 5-cycle will also be called a "pentagon".

(d) Two or more given cycles in G are (pairwise) "disjoint" (from each other) if no two of them have any vertex in common.

Definition 2.4. Suppose G is a (not necessarily connected) graph that has only finitely many vertices, no loops, and no multiple edges, and each vertex of G has valence 1, 2, or 3. (The valences of the different vertices are not assumed to be be equal.)

(a) A "3-edge-decomposition" of G is a partition of $\mathcal{E}(G)$ into three classes of edges such that no two adjacent edges of G are in the same class. The set of all 3-edge-decompositions of G is denoted $ED(G)$. For a given $\delta \in ED(G)$ and a given pair of edges e_1 and e_2 of G, the notation $e_1 \sim e_2$ means that e_1 and e_2 belong to the same class (of the three classes) in the decomposition δ.

(b) An "edge-3-coloring" of G is a function $\gamma : \mathcal{E}(G) \to \{a, b, c\}$ (where a, b, and c are the nonzero elements of $\mathbf{Z}_2 \times \mathbf{Z}_2$ defined in (2.1)) such that $\gamma(e_1) \neq \gamma(e_2)$ for every pair of adjacent edges e_1 and e_2 of G. The set of all edge-3-colorings of G is denoted $EC(G)$. If the set $EC(G)$ is nonempty, then one says simply that the graph G is "edge-3-colorable" or "can be edge-3-colored."

An edge-3-coloring of G is sometimes simply called a "coloring" of G. The phrases "edge-3-colorable" and "can be edge-3-colored" are sometimes simply abbreviated "colorable" and "can be colored". If G cannot be (edge-3-)colored, it is often referred to simply as being "uncolorable" or "noncolorable".

Edge-3-colorings, of cubic graphs (Definition 2.5(b) below), were first studied by Peter Guthrie Tait in 1880 (see [Wi, Chapter 6]), and are often called "Tait colorings".

Definition 2.5. Suppose G is a (not necessarily connected) graph that has only finitely many vertices, no loops, and no multiple edges.

(a) This graph G is said to be "quasi-cubic" if every vertex of it is either 1-valent (univalent) or 3-valent.

(b) The graph G is said to be "cubic" if every vertex of it is 3-valent.

Thus if G is cubic, then it is quasi-cubic.

If G is a quasi-cubic graph in which no vertex is 3-valent (i.e. every vertex is univalent), then G trivially has just one edge or just a collection of isolated edges. Such graphs will ordinarily be explicitly excluded (as in the next Remark).

Remark 2.6. Suppose G is a (not necessarily connected) graph with finitely many vertices, no loops, and no multiple edges, and G is quasi-cubic. Suppose further that G has at least one 3-valent vertex v. Let e_1, e_2, and e_3 denote the three edges connected to v. Then for every $\delta \in ED(G)$ (if one exists) there exists exactly one coloring $\gamma \in EC(G)$ such that (i) the equalities

$$\gamma(e_1) = a, \quad \gamma(e_2) = b, \quad \gamma(e_3) = c \tag{2.2}$$

hold (again see (2.1)) and (ii) the edges in any of the three classes in the decomposition δ are assigned the same color by γ. In fact this induces a one-to-one correspondence between $ED(G)$ and the set of all $\gamma \in EC(G)$ such that (2.2) holds. Hence

$$\text{card}\, ED(G) \;=\; \text{card}\, \{\gamma \in EC(G) : (2.2) \text{ holds}\}. \tag{2.3}$$

Since there are 6 permutations of the three colors a, b, and c, it follows that each $\delta \in ED(G)$ induces exactly 6 colorings $\gamma \in EC(G)$ such that the edges in any of the three classes in δ are assigned the same color by γ.

To summarize: If G is a (not necessarily connected) graph with finitely many vertices, no loops, and no multiple edges, and G is quasi-cubic and has at least one 3-valent vertex, then

$$\text{card}\, EC(G) \;=\; 6 \cdot \text{card}\, ED(G). \tag{2.4}$$

(Of course this holds trivially, with both sides being 0, in the case where no 3-edge-decomposition of G exists, i.e. when G cannot be colored.)

Definition 2.7. A graph G (not necessarily quasi-cubic) is said to be "simple" if it has only finitely many vertices, has no loops and no multiple edges, and is *connected*.

Remark. If a graph is both simple and quasi-cubic and has at least one 3-valent vertex, then trivially no edge of it can be connected to two univalent vertices. That is to be tacitly kept in mind in what follows.

Definition 2.8. Suppose G is a simple graph (not necessarily quasi-cubic).

(a) If Q is a proper subset of $\mathcal{V}(G)$, then $G - Q$ denotes the graph that one obtains from G by deleting every vertex in Q and every edge that is connected to at least one vertex in Q. (All vertices in the set $\mathcal{V}(G) - Q$ are left intact, even the ones that are endpoints of edges that are removed.)

(b) If S is any subset of $\mathcal{E}(G)$, then $G - S$ denotes the graph that one obtains from G by deleting every edge in S (but not deleting any vertex). That is, $G - S$ is the graph that consists of (i) all vertices of G and (ii) all edges of G except the ones in S.

(c) If the graph G has at least one cycle, then the "girth" of G is the smallest integer n (≥ 3) such that G has an n-cycle.

(If G has no cycles, i.e. G is a tree, then its "girth" is not defined.)

(d) Suppose the graph G has at least one pair of disjoint cycles (recall Definition 2.3(d)). For a given integer $n \geq 2$, the graph G is said to be (at least) "cyclically n-edge-connected" if the following holds: If C_1 and C_2 are any two disjoint cycles in G, and S is any subset of $\mathcal{E}(G)$ with card $S \leq n - 1$ such that S has none of the edges in the cycles C_1 and C_2 (edges in S may have *endpoint vertices* in C_1 and/or C_2), then there exists in the graph $G - S$ a path from (a vertex of) C_1 to (a vertex of) C_2.

(For a simple graph G that does not have any pair of disjoint cycles, the notion of "cyclically n-edge-connected" is not defined.)

In graph theory, there are other notions of "n-connectedness" (for a given positive integer n); but the one in (d) ("cyclically n-edge-connected") is the one that is most directly relevant to the material in this survey, and is the one that we shall stick with.

(e) The following two facts are well known and elementary to verify, and they are relevant to the next definition. (i) If a G is a simple graph whose vertices all have valence at least 2, then G has at least one cycle. (ii) If G is a simple *cubic* graph whose girth is at least 5, then it has at least one pair of disjoint cycles — in fact, for any cycle C with the *minimum* number of vertices (namely the girth), there exists another cycle that is disjoint from C. (For (ii), under the conditions on G and C, no vertex of $G - \mathcal{V}(C)$ can be adjacent (in G) to more than one vertex of C; and as a consequence, (i) implies that $G - \mathcal{V}(C)$ has a cycle.)

Definition 2.9 (snarks). Suppose G is a simple cubic graph. Then G is a "snark" if it has all of the following three properties:
(i) the girth of G is at least 5,
(ii) G is (at least) cyclically 4-edge-connected, and
(iii) G cannot be edge-3-colored.

Remark. Here the "primary" property is (iii). The "secondary" properties (i) and (ii) are standard restrictions, and in essence their purpose is to exclude noncolorable simple cubic graphs whose noncolorability can be "reduced" in a trivial way to that of some smaller noncolorable simple cubic graph. The term "snark" was suggested (borrowed from Lewis Carroll) by Martin Gardner [Ga] as a convenient term to describe the class of graphs that had been studied in the paper published a year earlier by Rufus Isaacs [Is] — the graphs meeting the conditions in Definition 2.9.

Remark 2.10 (the Petersen graph). (i) The classic "smallest" example of a snark is the Petersen graph, henceforth denoted $\mathcal{P}$, with ten vertices u_i and v_i, $i \in \{0, 1, 2, 3, 4\}$

and fifteen edges (u_i, u_{i+1}), (u_i, v_i), and (v_i, v_{i+2}), $i \in \{0, 1, 2, 3, 4\}$ (the addition in the subscripts is modulo 5).

(The proof that the Petersen graph $\mathcal{P}$ is a snark, is straightforward. One can show that it has no 3-cycles or 4-cycles, and that any two *disjoint* cycles must be two pentagons (5-cycles) directly connected by the remaining 5 edges. To see that $\mathcal{P}$ cannot be edge-3-colored, one can first identify possible color patterns for the edges (u_i, u_{i+1}), $i \in \{0, 1, 2, 3, 4\}$ — using symmetries and permutations of colors, one can reduce to basically just one color pattern — and one can then see that such a color pattern dictates the colors for the five edges (u_i, v_i), and those colors in turn lead to a contradiction when one tries to assign colors to the remaining edges (v_i, v_{i+2}).)

(ii) The Petersen graph $\mathcal{P}$ is loaded with symmetries. Here is just a small part of that story: If P_1 and P_2 are any two pentagons (5-cycles) in $\mathcal{P}$ (those two pentagons may be identical or overlapping or disjoint), and one maps P_1 onto P_2 in any of the (ten) possible ways, then that mapping extends to a unique automorphism of $\mathcal{P}$. Also, every edge of $\mathcal{P}$ belongs to a pentagon (in fact, to four pentagons). It follows that any edge of $\mathcal{P}$ can be mapped to any other edge as part of some automorphism of $\mathcal{P}$.

In papers such as those of Isaacs [Is] and Kochol [Ko1], methods have been devised to "combine" two or more snarks in order to form a "bigger" snark — and thereby to create recursively, starting with (say) the Petersen graph, infinite classes of arbitrarily large snarks. More on that later in this survey.

Terms such as "a Petersen graph" (singular) or "Petersen graphs" (plural) will be used for referring to one or more graphs that are isomorphic to $\mathcal{P}$.

Definition 2.11. Suppose H is a simple quasi-cubic graph that has at least one 3-valent vertex and can be edge-3-colored.

(a) Suppose $\gamma \in EC(H)$. Suppose K is a subgraph of H such that (i) K is connected, (ii) γ assigns just two colors, say x and y, to the edges in K, and (iii) all edges of H that are connected to (vertices of) K but are themselves not in K, are assigned the third color, say z. Then K is called a "Kempe chain" for the edge-3-coloring γ. (That is, a Kempe chain is a "maximal" subgraph K with properties (i) and (ii).)

(b) In (a), if the two colors assigned to the edges in K are x and y, then K is also called more specifically an xy-Kempe chain (for the coloring γ).

(Note that for a given coloring γ and two given colors x and y, any two different xy-Kempe chains must be disjoint from each other. Of course an xy-Kempe chain can intersect an xz-Kempe chain, where z is the third color.)

(c) If a Kempe chain (for a given $\gamma \in EC(H)$) is a cycle, then it is called a "Kempe cycle" (or an "xy-Kempe cycle" if the two colors are x and y).

(d) *Remark.* It is easy to see that for a given Kempe chain K (for a given $\gamma \in EC(H)$), either (i) K is simply a path with two distinct endpoints, each of which is univalent, or (ii) K is a (Kempe) cycle. If H is cubic, then every Kempe chain must be a (Kempe) cycle.

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(e) *Remark.* Suppose γ is an edge-3-coloring of H, x and y are distinct colors (in the set $\{a, b, c\}$ of colors from (2.1)), and K is an xy-Kempe chain for γ.

(i) Suppose one interchanges the colors x and y on just the Kempe chain K. That is, suppose one defines the mapping $\mu : \mathcal{E}(H) \to \{a, b, c\}$ as follows:

$$
\mu(e) := \begin{cases}
x & \text{if } e \in \mathcal{E}(K) \text{ and } \gamma(e) = y \\
y & \text{if } e \in \mathcal{E}(K) \text{ and } \gamma(e) = x \\
\gamma(e) & \text{for all edges } e \text{ of } H \text{ except the ones in } K.
\end{cases}
$$

Then μ is an edge-3-coloring of H, and also K is an xy-Kempe chain for μ (as well as for γ).

(ii) If one now starts with this new coloring μ and one (again) interchanges the colors x and y on K, then one obtains the original coloring γ.

Remark 2.12. Here is a quick review of some well known further elementary facts. Suppose G is a simple quasi-cubic graph that has at least one 3-valent vertex.

(a) Consider the ordered pairs (v, e) where v is a vertex of G and e is an edge of G connected to v. The number of such ordered pairs is even (since each edge generates two of them). Also, the total (even) number of such ordered pairs is the sum of the valences (1 or 3) of the vertices. It follows that *the number of vertices of G is even* (since otherwise the sum of the valences of the vertices would be odd, not even).

(b) Now suppose $\gamma \in EC(G)$, and x is one of the colors (a, b, or c — see (2.1)). The number of vertices connected to an edge colored x (by γ) is even (since each edge colored x is connected to two such vertices). Hence the number of vertices *not* connected to an edge colored x is even, and of course such vertices must be univalent.

(c) As a trivial consequence, for a given $\gamma \in EC(G)$, the number of univalent vertices that *are* connected to an edge colored x is going to be
(i) even for each color x ($\in \{a, b, c\}$) if the total number of univalent vertices is even,
(ii) odd for each color x ($\in \{a, b, c\}$) if the total number of univalent vertices is odd.
That fact is known as the "Parity Lemma".

(d) As a key special case of remark (c)(ii) above, if exactly five edges are connected to (different) univalent vertices, then for a given $\gamma \in EC(G)$, one color is given to exactly three of those five edges, and the other two colors are each given to exactly one of those five edges.

(e) Quasi-cubic graphs and the "Parity Lemma" have always played a ubiquitous role in the study of snarks. The "Parity Lemma" has the following (equivalent) well known classic formulation in terms of the non-zero elements of $\mathbf{Z}_2 \times \mathbf{Z}_2$ (see (2.1)):

Lemma 2.13 (Parity Lemma). *Suppose H is a simple quasi-cubic graph which has at least one 3-valent vertex but is not cubic. Let $e_1, e_2, \ldots, e_n$ denote the edges of H that are each connected to a univalent vertex.*

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Suppose also that H is edge-3-colorable, and γ is an edge-3-coloring of H. Then

$$\sum_{i=1}^{n} \gamma(e_i) = 0. \tag{2.5}$$

(Here the addition is that of $\mathbf{Z}_2 \times \mathbf{Z}_2$ as in Definition 2.1; and the right hand side is of course the zero element (0,0) of $\mathbf{Z}_2 \times \mathbf{Z}_2$ as in (2.1).)

Remark. For a coloring γ of a colorable simple cubic graph G, if $\{e_1, e_2, \ldots, e_n\}$ is (in the sense of inclusion) a "minimal cut set" of edges of G, then (2.5) holds. To see that, one can simply apply Lemma 2.13 itself to either one of the two disjoint simple quasi-cubic graphs that one obtains from G by (say) "cutting each of the edges e_i in half" (and inserting a vertex at the end of each of the resulting "strands"). (Here of course the term "cut set" means that the graph $G - \{e_1, \ldots, e_n\}$ is not connected.)

Kochol [Ko2] gives a generalization of Lemma 2.13 and uses it in the study of "graph coloring" problems in more general contexts than just cubic graphs. That will not be treated further here.

Remark 2.14 (the flower snarks). Here we just point out a special class of snarks known as the "flower snarks":

Suppose n is an *odd* integer such that $n \geq 5$. Let J_n denote the simple cubic graph that consists of $4n$ vertices t_k, u_k, v_k, w_k, $k \in \{0, 1, 2, \ldots, n-1\}$ and the following $6n$ edges: (t_k, t_{k+1}), (u_k, v_{k+1}), (v_k, u_{k+1}), (w_k, t_k), (w_k, u_k), (w_k, v_k), $k \in \{0, 1, 2, \ldots, n-1\}$. Here addition in the subscripts is modulo n; for example, for $k = n - 1$, $t_{k+1} := t_0$.

Isaacs [Is, Theorem 4.1.1] showed that for each odd integer $n \geq 5$, this graph J_n — as Isaacs [Is] himself called it — is a snark. (We shall not give the argument here.) This class J_n, $n \in \{5, 7, 9, 11, \ldots\}$ was one of the large classes of snarks that was brought to light by Isaacs [Is]. These snarks J_n, $n \in \{5, 7, 9, 11, \ldots\}$ were later referred to by Gardner [Ga] as the "flower snarks", and that name has become customary since then for these particular snarks. In what follows, the flower snarks will occasionally be alluded to briefly. It will just be mentioned here in passing that the flower snarks have special properties and a special fascination and are a topic of interest in their own right. For example, Tinsley and Watkins [TW] computed the (topological) genus of each of the flower snarks.

Isaacs [Is] (and Gardner [Ga]) included $n = 3$ in the definition above. Technically, J_3 is not a snark, because it has a "triangle" (a 3-cycle); but (as Isaacs [Is] noted) if one "contracts" that triangle in J_3 to a single vertex, one obtains the Petersen graph.

(To match the definition of the graphs J_n given above to the pictures of those graphs in [Is, p. 233] and [Ga, p. 128], first decipher the "Remark" (both paragraphs of it) in [Is, p. 234].)

3. Results of Kászonyi

Throughout the rest of this survey paper, it is tacitly understood that theorems or proofs or informal comments that involve "snarks", often trivially carry over to — and in some cases were originally formulated for — some broader classes of noncolorable simple cubic graphs that were allowed to satisfy less stringent "secondary" conditions than conditions (i) and (ii) in Definition 2.9. As a convenient formality, our discussion of noncolorable simple cubic graphs will be confined to snarks (i.e. satisfying all three conditions in Definition 2.9); we shall thereby avoid having to occasionally bother with some trivial extra technicalities.

Section 3 here is an exposition of some of the work of Kászonyi [Ká1, Ká2, Ká3] that pertains to snarks and also involves a closely related topic: edge-3-colorable simple cubic graphs with "orthogonal edges" (Definition 3.1 below). The connection between those two topics is given as part of Theorem 3.3 below. In its entirety, Theorem 3.3 itself, a result of Kászonyi [Ká2, Ká3], is the "backbone" of this entire survey paper.

Definition 3.1 (Kászonyi [Ká1, Ká2, Ká3]).   Suppose H is a simple cubic graph which can be edge-3-colored. Suppose d_1 and d_2 are edges of H. Those edges d_1 and d_2 are said to be "orthogonal" (in H) if there does not exist $\gamma \in EC(H)$ for which d_1 and d_2 are edges in the same (two-color) Kempe cycle.

Remark.   Of course in the context of this definition, the "orthogonal" edges d_1 and d_2 cannot be adjacent. (If d_1 and d_2 were adjacent, then for a given edge-3-coloring γ of H, if one lets x and y denote the colors of d_1 and d_2 respectively, those two edges would belong to the same xy-Kempe cycle for γ, contradicting the definition of "orthogonal edges".)

Notations 3.2.   Suppose G is a simple cubic graph that has girth at least 4 and is at least cyclically 2-edge-connected, and $e = (u, v)$ is an edge of G. Let t_i, $i \in \{1, 2\}$ denote the two vertices $\neq v$ that are adjacent to u in G; and let w_i, $i \in \{1, 2\}$ denote the two vertices $\neq u$ that are adjacent to v in G. (By the assumptions on G, those four vertices t_1, t_2, w_1, and w_2 are distinct, and G does not have an edge of either the form (t_1, t_2) or (w_1, w_2).)

Let G_e denote the simple cubic graph that consists of the graph $G - \{u, v\}$ and the two new edges $d_1 := (t_1, t_2)$ and $d_2 := (w_1, w_2)$.

That is, to obtain G_e, one deletes from G the two vertices u and v and all five edges (including e) connected to them, and one then inserts the two new edges d_1 and d_2. The simple cubic graph G_e is conformal to the graph $G - \{e\}$. (The fact that G_e is simple and cubic, is an elementary consequence of the assumptions here on G itself.)

In Notations 3.2, the graph G is not assumed to be a snark. However, apart from Problem 9 in Section 7.1, all of the applications of Notations 3.2 in this survey paper will explicitly involve the case where G is assumed to be a snark.

Theorem 3.3 (Kászonyi). *(A) Suppose H is a simple cubic graph which can be edge-3-colored, and d_1 and d_2 are orthogonal edges of H (see Definition 3.1). Then there exists a positive integer J such that the following three statements (i), (ii), (iii) hold:*
(i) card $ED(H) = 3J$ (and hence card $EC(H) = 18J$).
(ii) card $\{\delta \in ED(H) : d_1 \sim d_2\} \ = \ J$.
(iii) If x and y each $\in \{a, b, c\}$ (x and y may be the same or different), then card $\{\gamma \in EC(H) : \gamma(d_1) = x$ and $\gamma(d_2) = y\} \ = \ 2J$.

(B) Suppose G is a snark and e is an edge of G. Let d_1 and d_2 be the edges of G_e specified in Notations 3.2. If the (simple cubic) graph G_e is edge-3-colorable, then d_1 and d_2 are orthogonal edges of G_e.

(C) Suppose G is a snark and e is an edge of G. Let d_1 and d_2 be the edges of G_e specified in Notations 3.2. Then there exists a nonnegative integer L such that the following three statements (1), (2), (3) hold:
(1) card $ED(G_e) = 3L$ (and hence card $EC(G_e) = 18L$).
(2) card $\{\delta \in ED(G_e) : d_1 \sim d_2\} \ = \ L$.
(3) If x and y each $\in \{a, b, c\}$ (x and y may be the same or different), then card $\{\gamma \in EC(G_e) : \gamma(d_1) = x$ and $\gamma(d_2) = y\} \ = \ 2L$.

This theorem, and its proof given below, are due to Kászonyi [Ká2, Ká3]. In the proof of statement (A) given below, there are three special classes of edge-3-colorings of H that are called (in the proof below) Q_a, Q_b, Q_c (where a, b, c are the "colors" from (2.1)). The one-to-one correspondences (involving just the interchange of the colors on particular Kempe cycles) between those three classes, as spelled out in the proof below, were pointed out by Kászonyi [Ká3, p. 35, the paragraph after Theorem 4.3 (together with p. 28, Section 1.15)] (in a somewhat cryptic manner — see Section 7.2(D) in Section 7). Those one-to-one correspondences together give statement (A). The proof of statement (B) given below is taken from [Ká2, p. 125, lines 8-24]. (Kászonyi's own formulation and argument there were technically slightly more general, in the sense that the simple cubic graph G, while assumed to be noncolorable, was allowed to satisfy somewhat less stringent "secondary" conditions than conditions (i) and (ii) in Definition 2.9. The term "snark" as in Definition 2.9 had not yet been coined or codified.) Statement (C) is (aside from the trivial case where G_e cannot be edge-3-colored) simply an application of statements (A) and (B).

Proof. Proof of statement (A). Let v be one of the end-point vertices of the edge d_1, and let e_1 and e_2 be the other two edges (besides d_1) that are connected to v. By the Remark after Definition 3.1, neither e_1 nor e_2 is the edge d_2.

As described in Remark 2.6, every $\delta \in ED(H)$ induces a unique $\gamma \in EC(H)$ such that

$$\gamma(d_1) = a, \quad \gamma(e_1) = b, \quad \text{and} \quad \gamma(e_2) = c. \tag{3.1}$$

For each $x \in \{a, b, c\}$, let Q_x denote the set of all $\gamma \in EC(H)$ such that (3.1) holds and $\gamma(d_2) = x$.

At this point, we make no assumptions on whether or not the set Q_x, for a given color x, may be empty. Accordingly, for now, definitions and arguments below involving Q_x are allowed to be "vacuous".

The assumption (in statement (A)) that the edges d_1 and d_2 are orthogonal, has the following elementary consequence: For any γ in either Q_a or Q_b (if such a γ exists), if one interchanges the colors a and b on the ab-Kempe cycle containing the edge d_2, that will not change the colors (in (3.1)) of any of the edges d_1, e_1, or e_2. Thus one has (for now, possibly vacuous) mappings $M : Q_a \to Q_b$ and $M^* : Q_b \to Q_a$ defined as follows: For each $\gamma \in Q_a$ (resp. $\gamma \in Q_b$), let $M\gamma$ (resp. $M^*\gamma$) denote the element of Q_b (resp. of Q_a) which is obtained from γ by the interchanging of the colors a and b on the ab-Kempe cycle (for γ) containing the edge d_2. By a trivial argument (recall Remark (e)(i)(ii) in Definition 2.11), the mappings M and M^* are inverses of each other: $M^*M\gamma = \gamma$ for $\gamma \in Q_a$, and $MM^*\gamma = \gamma$ for $\gamma \in Q_b$. It follows that M is one-to-one and onto, as a mapping from Q_a to Q_b. (And of course an analogous comment holds for M^*.) Hence $\mathrm{card}\, Q_a = \mathrm{card}\, Q_b$.

By an exactly analogous argument, $\mathrm{card}\, Q_a = \mathrm{card}\, Q_c$. Thus

$$\mathrm{card}\, Q_a \;=\; \mathrm{card}\, Q_b \;=\; \mathrm{card}\, Q_c. \tag{3.2}$$

Define the nonnegative integer J by

$$J \;:=\; \mathrm{card}\, Q_a. \tag{3.3}$$

In the same manner as in Remark 2.6, one has a one-to-one correspondence between the sets $\{\delta \in ED(H) : d_1 \sim d_2\}$ and Q_a. (Every $\delta \in ED(H)$ such that $d_1 \sim d_2$, trivially induces a unique $\gamma \in Q_a$.) Hence by (3.3), the equality in sub-statement (ii) in statement (A) holds.

Also, precisely as in eq. (2.3) in Remark 2.6,

$$\mathrm{card}\, ED(H) \;=\; \mathrm{card}\, \{\gamma \in EC(H) : (3.1) \text{ holds}\}. \tag{3.4}$$

Now the set in the right hand side of (3.4) is simply $Q_a \cup Q_b \cup Q_c$. Also, trivially by definition, those sets Q_a, Q_b, and Q_c are (pairwise) disjoint. Hence by (3.2), (3.3), and (3.4), $\mathrm{card}\, ED(H) = 3J$. Hence by eq. (2.4) in Remark 2.6, sub-statement (i) in statement (A) holds. Also, by the hypothesis of statement (A), the set $EC(H)$ (or $ED(H)$) is nonempty. Hence by sub-statement (i) itself in statement (A), the integer J is positive.

Hence, in the proof of statement (A), all that now remains is to verify sub-statement (iii), which is just an automatic, trivial by-product of sub-statements (i) and (ii). Here are the details:

Proof of sub-statement (iii). First, any $\delta \in ED(H)$ such that $d_1 \sim d_2$, induces six colorings $\gamma \in EC(H)$ such that $\gamma(d_1) = \gamma(d_2)$ (with the edges in any of the three classes in δ being assigned the same color by γ). Hence by sub-statement (ii),

$$\mathrm{card}\, \{\gamma \in EC(H) : \gamma(d_1) = \gamma(d_2)\} \;=\; 6J. \tag{3.5}$$

By trivial permutations of colors, one can show that for the three colors $x \in \{a, b, c\}$, the numbers $\mathrm{card}\, \{\gamma \in EC(H) : \gamma(d_1) = \gamma(d_2) = x\}$ are equal, and hence by (3.5) they must each be equal to $2J$. Thus sub-statement (iii) holds in the case where $x = y$.

Next, by (3.5) and sub-statement (i) (in statement (A)),

$$\operatorname{card}\{\gamma \in EC(H) : \gamma(d_1) \neq \gamma(d_2)\} \;=\; 12J. \tag{3.6}$$

By trivial permutations of colors, one can show that for the six permutations (x, y) of two distinct elements of $\{a, b, c\}$, the numbers $\operatorname{card}\{\gamma \in EC(H) : \gamma(d_1) = x \text{ and } \gamma(d_2) = y\}$ are equal, and hence by (3.6) they must each be equal to $2J$. Thus sub-statement (iii) holds for the case $x \neq y$. That completes the proof of sub-statement (iii), and of statement (A).

Proof of statement (B). Suppose G_e can be edge-3-colored. Suppose the edges d_1 and d_2 are *not* orthogonal. We shall aim for a contradiction. The argument will be given here in a somewhat informal way, but that should not cause any confusion. We shall make detailed use of the symbols in Notations 3.2.

Let γ be an edge-3-coloring of G_e such that for two given colors, say a and b, the edges d_1 and d_2 belong to the same ab-Kempe chain K.

Let us "reinsert" the vertices u and v (see Notations 3.2) into the middle of the edges d_1 and d_2 respectively, thereby "re-obtaining" the graph $G - \{e\}$ (which has two "2-valent" vertices u and v, and whose other vertices are still 3-valent).

Those two "newly reinserted" vertices u and v split the above-mentioned ab-Kempe chain K for the coloring γ into two pieces. On either one of those two pieces (but not the other), interchange the colors a and b. Thereby one obtains an edge-3-coloring of the graph $G - \{e\}$ in which (i) no two adjacent edges have the same color and (ii) the two edges connected to the vertex u (respectively to v) have (in either order) the colors a and b.

Now "reinsert" the edge e to "re-obtain" the original snark G, and assign to the "newly re-inserted" edge e the color c. One thereby obtains an edge-3-coloring of the snark G, contradicting the definition of "snark".

Thus a contradiction has occurred, and hence the edges d_1 and d_2 must be orthogonal after all. That completes the proof of statement (B).

Proof of statement (C). In the case where the graph G_e itself cannot be edge-3-colored, statement (C) (all parts of it) hold trivially with $L = 0$. In the case where the graph G_e can be edge-3-colored, statement (C) follows immediately from statements (A) and (B) (with the graph H in statement (A) being G_e in statement (C), and with the integer L in statement (C) being the integer J in statement (A)). That completes the proof of statement (C), and of Theorem 3.3. ////

Definition 3.4. Refer to Theorem 3.3(C)(1). For any snark G and any edge e of G, define the nonnegative integer $\psi(G, e)$ — the "Kászonyi number" of G and e — as follows:

$$\psi(G, e) := (1/3) \cdot \operatorname{card} ED(G_e) . \tag{3.7}$$

That is, $\psi(G, e)$ denotes the nonnegative integer L such that $\operatorname{card} ED(G_e) = 3L$ (i.e. such that $\operatorname{card} EC(G_e) = 18L$).

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Remark. If G is a snark, and the graph H is an isomorphic copy of G, then H is also a snark; and if also e is any edge of G, and e^* is the edge of H which is the "image" of e under a given isomorphism from G to H, then $\psi(H, e^*) = \psi(G, e)$. The argument is elementary.

The notation $\psi(G, e)$, used in [Br2], implicitly comes from the result of Kászonyi [Ká2, Ká3] given in Theorem 3.3(C) above. It will be used throughout the rest of this paper.

Theorem 3.5 (Kászonyi). *For the Petersen graph $\mathcal{P}$ (see Remark 2.10) and any edge e of $\mathcal{P}$, one has that $\psi(\mathcal{P}, e) = 1$.*

This theorem is due to Kászonyi [Ká1, Ká2]. Kászonyi [Ká2, p. 126, the Remark] pointed out (in a slightly cryptic manner) that for the Petersen graph $\mathcal{P}$ and an edge e of $\mathcal{P}$, the graph $\mathcal{P}_e$ is (isomorphic to) a certain graph consisting of an "8-vertex wheel with four 'rim-to-rim spokes going through the hub'" (with no vertex at the "hub"). Slightly earlier, Kászonyi [Ká1, pp. 81-82] had already shown that for that particular graph (the "8-vertex wheel with rim-to-rim spokes"), there are exactly three 3-edge-decompositions. Thereby (see (3.7)) Theorem 3.5 was established.

The proof given below is a slightly shortened version of Kászonyi's argument, using the terminology in Definition 3.4 and taking advantage of statement (C)(3) in Theorem 3.3.

Proof. By Remark 2.10(ii) and the Remark after Definition 3.4, it suffices to carry out the argument for any one particular edge e of the Petersen graph $\mathcal{P}$. Referring to the formulation of the Petersen graph $\mathcal{P}$ in Remark 2.10(i), we shall consider the edge $e := (u_2, u_3)$ there, and we shall express the graph $W := \mathcal{P}_e$ as an "8-vertex wheel with rim-to-rim spokes" (Kászonyi's [Ká1, Ká2] way of portraying $\mathcal{P}_e$) as follows: The eight vertices of $W = \mathcal{P}_e$ will be relabeled as $t_i, i \in \{0, 1, 2, \ldots, 7\}$, with (see the notations in Remark 2.10(i)) $t_0 := u_4$, $t_1 := u_0$, $t_2 := u_1$, $t_3 := v_1$, $t_4 := v_3$, $t_5 := v_0$, $t_6 := v_2$, and $t_7 := v_4$. Then the twelve edges of W fall into two classes: the eight "wheel edges" $\epsilon_i := (t_i, t_{i+1})$, $i \in \{0, 1, \ldots, 7\}$ (with addition mod 8 — thus $\epsilon_7 := (t_7, t_0)$); and the four "spoke edges" $f_i := (t_i, t_{i+4})$, $i \in \{0, 1, 2, 3\}$. In the terminology of Notations 3.2, the edges d_1 and d_2 are (say) respectively

$$d_1 := (u_4, v_3) = (t_0, t_4) = f_0 \quad \text{and}$$
$$d_2 := (u_1, v_2) = (t_2, t_6) = f_2. \tag{3.8}$$

Referring to (3.7), our task is to show that card $ED(W) = 3$. Referring to (3.8) and Theorem 3.3(C)(1)(3), one has that it suffices to show that

$$\text{card} \{\gamma \in EC(W) : \gamma(f_0) = \gamma(f_2) = a\} = 2. \tag{3.9}$$

Let us count the ways of constructing colorings γ of W such that $\gamma(f_0) = \gamma(f_2) = a$. Each of the eight "wheel edges" ϵ_i, $i \in \{0, 1, \ldots, 7\}$ is connected to one of the vertices

t_0, t_2, t_4, or t_6, the endpoints of the edges f_0 and f_2. Hence those eight "wheel edges" must be colored alternately b or c. There are two ways of doing that, depending on (say) which color (b or c) is assigned to the edge ϵ_0. Either way, that forces one to complete the coloring γ of W by assigning the color a to the remaining two "spoke edges" f_1 and f_3. Thus (3.9) holds. That completes the proof of Theorem 3.5.   ////

The rest of the material here in Section 3 is somewhat peripheral to the main theme of this survey paper, and is included here only to round out the picture a little. Theorem 3.8 below is of independent interest.

Notations 3.6. Suppose H is a simple cubic graph which can be edge-3-colored, and d_1 and d_2 are orthogonal edges of H (see Definition 3.1). Let $\mathcal{Q}(H, d_1, d_2)$ denote the set of all subgraphs Λ of H with the following four properties:
(i) Λ is the union of one or more (pairwise) disjoint cycles in H.
(ii) Each cycle in Λ has an even number of edges.
(iii) Λ contains every vertex of H.
(iv) Neither d_1 nor d_2 is an edge of Λ.
   For each $\Lambda \in \mathcal{Q}(H, d_1, d_2)$, let $\mathcal{N}(\Lambda)$ denote the number of cycles in Λ.

Theorem 3.7 (Kászonyi). *(A) Suppose H is a simple cubic graph which can be edge-3-colored, and d_1 and d_2 are orthogonal edges of H (see Definition 3.1). Then (see Notations 3.6 above)*

$$\operatorname{card} ED(H) \;=\; (3/2) \cdot \sum_{\Lambda \in \mathcal{Q}(H, d(1), d(2))} 2^{\mathcal{N}(\Lambda)} \qquad (3.10)$$

(where d_1 and d_2 are written here as $d(1)$ and $d(2)$ for typographical convenience).
   (B) If G is a snark, e is an edge of G, and the (simple cubic) graph G_e has no Hamiltonian cycle, then the nonnegative integer $\psi(G, e)$ is even.

Theorem 3.7, and its proof given below, are due to Kászonyi [Ká3, p. 35, Section 4.2]. Statement (A) was shown there — together with some other closely related information — in an indirect, somewhat cryptic form (see Section 7.2(D) in Section 7). Statement (B) is simply a special case of statement (A).

Proof. Proof of statement (A). Refer to Theorem 3.3(A)(i)(iii) (and to Definition 2.1). To prove (3.10) and thereby statement (A), it suffices to show that

$$\operatorname{card} \{\gamma \in EC(H) : \gamma(d_1) = \gamma(d_2) = a\} \;=\; \sum_{\Lambda \in \mathcal{Q}(H, d(1), d(2))} 2^{\mathcal{N}(\Lambda)} . \qquad (3.11)$$

For any $\gamma \in EC(H)$ such that $\gamma(d_1) = \gamma(d_2) = a$, the set of edges that are colored b or c (by γ), together with their endpoints, form a subgraph Λ in the class $\mathcal{Q}(H, d_1, d_2)$. Hence to count the colorings $\gamma \in EC(H)$ such that $\gamma(d_1) = \gamma(d_2) = a$, it suffices to take the sum, over all $\Lambda \in \mathcal{Q}(H, d_1, d_2)$, of the number of colorings γ (with $\gamma(d_1) = \gamma(d_2) = a$)

whose bc-Kempe cycles together form the graph Λ. For each such Λ, there are exactly $2^{\mathcal{N}(\Lambda)}$ such colorings, since for each cycle in Λ, there are exactly two ways of assigning the alternating colors b and c. Thus (3.11) holds. That completes the proof of statement (A).

Proof of statement (B). In the case where the graph G_e itself cannot be edge-3-colored, statement (B) is trivial, with $\psi(G, e) = 0$. Therefore, assume G_e can be colored.

Let $H := G_e$, and let the edges d_1 and d_2 be as in Notations 3.2. By Theorem 3.3(B), those two edges d_1 and d_2 are orthogonal. If $\Lambda \in \mathcal{Q}(H, d_1, d_2)$, then it must have at least two cycles (since otherwise Γ itself would be a Hamiltonian cycle, contradicting the hypothesis of statement (B)) — that is $\mathcal{N}(\Lambda) \geq 2$ — and hence $2^{\mathcal{N}(\Lambda)}$ is a multiple of 4. Hence the right side of (3.10) is a multiple of 6. Hence by (3.10) itself and (3.7), the integer $\psi(G, e) = (1/3) \operatorname{card} ED(H)$ is even. That completes the proof of statement (B), and of Theorem 3.7. ////

Theorem 3.8 (Kászonyi). *Suppose H is a simple cubic graph which can be edge-3-colored, and it has at least one pair of orthogonal edges. Suppose further that $\operatorname{card} ED(H) = 3$ (the smallest possible number under the assumptions here — see Theorem 3.3(A)). Then H is nonplanar.*

This theorem is due to Kászonyi [Ká2, Theorem 6]. This theorem and its proof are of intrinsic interest in their own right, but will not be needed anywhere else in this survey paper. The proof is somewhat long and complicated and will not be repeated here.

4. Pentagons

Recall from Definition 2.3(c) that a 5-cycle is also called a "pentagon". Not all snarks have a pentagon. For example, the flower snarks J_n, $n \in \{7, 9, 11, \ldots\}$ in Remark 2.14 do not have a pentagon. Kochol [Ko1] constructed snarks with arbitrarily large girth. In contrast, the Petersen graph (see Remark 2.10) has twelve pentagons; and the flower snark J_5 (again see Remark 2.14) has exactly one. Section 4 here will be narrowly focused on certain facts involving pentagons in snarks.

In particular, if G is a snark and P is a pentagon (if one exists) in G, then the numbers $\psi(G, e)$, $e \in \mathcal{E}(P)$ are equal. That and some related information will be given in Theorems 4.5 and 4.8 below, after some background information. Remarks 4.2 and 4.3 below are well known and quite trivial, and their proofs will be omitted.

Convention 4.1. Here in Section 4, in notations such as v_k for vertices, the indices k will be taken as elements of the field $\mathbf{Z}_5$. The elements of $\mathbf{Z}_5$ will be denoted simply $0, 1, 2, 3, 4$, with addition and multiplication mod 5.

For example, in such a context one has for $k = 4$ that $k + 1 = 0$ and $2k = 3$.

Remark 4.2. Suppose x, y, and z are (in any "order") three distinct elements of $\mathbf{Z}_5$.

Consider the following three conditions:

(i) There exists $k \in \mathbf{Z}_5$ such that $\{x, y, z\} = \{k - 1, k, k + 1\}$.
(ii) There exists $k \in \mathbf{Z}_5$ such that $\{x, y, z\} = \{k - 2, k, k + 2\}$.
(ii′) There exists $j \in \mathbf{Z}_5$ such that $\{2x, 2y, 2z\} = \{j - 1, j, j + 1\}$.

Then by trivial arithmetic (mod 5), the following three statements hold:
(A) Exactly one of conditions (i), (ii) holds.
(B) Conditions (ii) and (ii′) are equivalent.
(C) Hence, exactly one of conditions (i), (ii′) holds.

Remark 4.3. Suppose G is a simple cubic graph with girth 5, and P is a pentagon in G with vertices v_i and edges (v_i, v_{i+1}), $i \in \mathbf{Z}_5$. The graph $G - \mathcal{E}(P)$ is quasi-cubic, and its univalent vertices are precisely the ones v_i, $i \in \mathbf{Z}_5$. For each $i \in \mathbf{Z}_5$, let u_i denote the vertex of G such that $\epsilon_i := (u_i, v_i)$ is an edge of $G - \mathcal{E}(P)$. The vertices u_i, $i \in \mathbf{Z}_5$ are distinct (a trivial consequence of the fact that G has no 3-cycles or 4-cycles). Suppose $G - \mathcal{E}(P)$ can be edge-3-colored, and $\gamma \in EC(G - \mathcal{E}(P))$.

(A) By the "Parity Lemma" (the special case of it in Remark 2.12(d)), one has that (i) for exactly three distinct indices $q, r, s \in \mathbf{Z}_5$, the equality $\gamma(\epsilon_q) = \gamma(\epsilon_r) = \gamma(\epsilon_s)$ holds, and (ii) for the other two indices t and u in $\mathbf{Z}_5$, the colors $\gamma(\epsilon_t)$, $\gamma(\epsilon_u)$, and $\gamma(\epsilon_q)$ $(= \gamma(\epsilon_r) = \gamma(\epsilon_s))$ are distinct.

(B) In (A), if $\{q, r, s\} = \{k - 1, k, k + 1\}$ for some $k \in \mathbf{Z}_5$ (that is, if q, r, s are in some order "consecutive" in $\mathbf{Z}_5$), then γ extends uniquely to an edge-3-coloring of the entire cubic graph G itself.

(C) In (A), if instead $\{q, r, s\} = \{k - 2, k, k + 2\}$ for some $k \in \mathbf{Z}_5$ (recall Remark 4.2(A)), then γ does not extend to an edge-3-coloring of G.

(D) Paragraph (A) above has the following equivalent formulation: For any given $\delta \in ED(G - \mathcal{E}(P))$, three of the edges ϵ_i, $i \in \mathbf{Z_5}$ belong to the same class (of the three classes in the decomposition δ), and the other two of the edges ϵ_i belong respectively to the other two classes.

Remark 4.4. Theorem 4.5 below will be motivated here by the following "historical" information:

(A) Consider a simple cubic *planar* graph H which has no "bridge" (an edge whose removal would split the remainder of H into two disjoint pieces). The Four Color Theorem states that there exists a "face-4-coloring" of H — a coloring Γ of the *faces* of H with (say) the four colors $0, a, b, c$ (the elements of $\mathbf{Z}_2 \times \mathbf{Z}_2$ — see (2.1)) such that no two contiguous faces are assigned the same color. It is well known that any face-4-coloring Γ of H with the colors $0, a, b, c$ induces an edge-3-coloring γ of H with the colors a, b, c in which for each edge e of H,

$$\gamma(e) = \Gamma(F_1) + \Gamma(F_2) \tag{4.1}$$

where F_1 and F_2 are the faces of H that share the edge e as a common border. (Recall that if x and y are any two distinct elements of $\mathbf{Z}_2 \times \mathbf{Z}_2$ — either one can be 0 — then $x + y \neq 0$.) It is also well known that conversely, for any $\gamma \in EC(H)$, any face F of H,

and any element $x \in \mathbf{Z}_2 \times \mathbf{Z}_2$, there exists a (unique) face-4-coloring Γ of H such that $\Gamma(F) = x$ and (4.1) holds for every edge e of H. These observations go back to the work of Peter Tait in 1880 alluded to after Definition 2.4; see [Wi, Chapter 6].

(B) In a paper published in 1879, Alfred Kempe gave an intended "proof" of the (then not yet proved) Four Color Theorem. The key facet of his argument was his claim that if H is a simple cubic planar graph with girth 5 and no "bridge", P is a pentagon in H, and Γ is a face-4-coloring of all faces of H except the face P, then (if necessary) after successive interchanges of colors along two different connected two-color regions (now known as "Kempe chains"), the coloring Γ can be extended to include the face P and thereby produce a face-4-coloring of (all of) H. Eleven years later, in a paper published in 1890, Percy Heawood showed with an example that Kempe's intended "proof" — in particular, the key facet of it described above — is invalid. (Heawood's example was of course not a counterexample to the Four Color Theorem itself; it just exposed a fatal flaw in Kempe's intended argument for it.) For a detailed description of Kempe's wrong "proof" and of Heawood's example, see [Wi, Chapters 5–7]. What (intentionally, by design) "goes wrong" in Heawood's example, will be referred to here informally as "Heawood's monkey wrench".

(C) Adapting the connection between face-4-colorings and edge-3-colorings (for simple cubic planar graphs with no "bridge") described in paragraph (A) above, one can transcribe the description of "Heawood's monkey wrench" from its original form (involving face-4-colorings) into a form involving edge-3-colorings. Such an "edge-3-coloring" version of "Heawood's monkey wrench" can then be adapted to some simple cubic graphs that are not planar.

(D) In particular, suppose G is a snark. Of course, by the Four Color Theorem itself (and paragraph (A) above), G is nonplanar. If P is a pentagon in G, then for the quasi-cubic graph $G - \mathcal{E}(P)$, if it is colorable, the "edge-3-coloring" version of "Heawood's monkey wrench" inevitably *has* to occur, over and over and over again, simply as a consequence of the fact that G itself cannot be edge-3-colored. The next theorem is based on that fact, and compiles some information that follows naturally from it.

Theorem 4.5. *Suppose G is a snark.*
(A) If H is a connected union of pentagons in G (or H is simply a single pentagon in G), then the numbers $\psi(G, e)$, $e \in \mathcal{E}(H)$ are equal.
(B) Suppose P is a pentagon in G with vertices v_i and edges (v_i, v_{i+1}), $i \in \mathbf{Z}_5$ (recall Convention 4.1). For each $i \in \mathbf{Z}_5$, let ϵ_i denote the edge of $G - \mathcal{E}(P)$ that is connected to the vertex v_i. Refer to conclusion (A) above, and also to Remark 4.3(D). Then the following two statements (i), (ii) hold:
(i) card $ED(G - \mathcal{E}(P)) = 5 \cdot \psi(G, e)$ where e is any edge of P.
(ii) For each $i \in \mathbf{Z}_5$, card $\{\delta \in ED(G - \mathcal{E}(P)) : \epsilon_{i-2} \sim \epsilon_i \sim \epsilon_{i+2}\} = \psi(G, e)$ where e is any edge of P.

In conclusion (A), it is understood that the union of *all* pentagons in G may be a disconnected subgraph of G, and that the common number $\psi(G, e)$ for the edges e of one

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connected component of that subgraph may be different from the common number $\psi(G, e)$ for the edges e of another connected component.

Of course in conclusion (B)(i)(ii), in the phrase "where e is any edge of P", e is of course (v_i, v_{i+1}) for some (any) $i \in \mathbf{Z}_5$.

Theorem 4.5, and its proof given below, are due to the author [Br1, Theorem 2 and Corollary 1].

It was also pointed out there (in [Br1, Theorem 2]) that in the context of Theorem 4.5(B) here, as a simple consequence of conclusion (ii), for any two non-adjacent edges d and f of P, one has that card $ED((G_d)_f) = \psi(G, e)$ for (any) $e \in \mathcal{E}(P)$.

The core idea in Theorem 4.5 and its proof is the repeated application of an edge-3-coloring version of "Heawood's monkey wrench" (see Remark 4.4(B)(C)(D) again). In the argument below, such a version of "Heawood's monkey wrench" plays its key role especially in Lemma 3(i)(ii) and its proof.

Proof. The proof of Theorem 4.5 will first proceed through a series of definitions, lemmas, etc. (numbered in order as 0, 1, 2, ..., 9) that will establish statement (B), together with statement (A) for the special case of a single pentagon. After that, at the very end, a final argument will be given to establish statement (A) in its full generality.

Throughout this proof, G is a given snark. It is assumed to have at least one pentagon, say P, (consistent with the hypotheses of each of statements (A) and (B)). However, the quasi-cubic graph $G - \mathcal{E}(P)$ is not assumed to be colorable; definitions and lemmas below are allowed to be "vacuous".

Proof of statement (B) (and of statement (A) for one pentagon). The proof will start with a "Context" that will provide the setting for the entire argument for statement (B) (and for statement (A) for one pentagon).

Context 0. Suppose P is a pentagon (5-cycle) in G. Let the vertices of P be denoted in order as v_0, v_1, v_2, v_3, v_4, with the edges of P being (v_i, v_{i+1}) (of course using Convention 4.1). For each $i \in \{0, 1, 2, 3, 4\}$, let ϵ_i denote the edge of $G - \mathcal{E}(P)$ that is connected to the vertex v_i, and let w_i denote the "other" endpoint vertex of ϵ_i, so that $\epsilon_i = (w_i, v_i)$. As a consequence of the snark G having no 3-cycles or 4-cycles (see Definition 2.9), the vertices w_i, $i \in \{0, 1, 2, 3, 4\}$ (as well as the vertices v_i, $i \in \{0, 1, 2, 3, 4\}$) are distinct.

Definition 1. Refer to Remark 4.3(A)(B)(C). For each $k \in \{0, 1, 2, 3, 4\}$ (recall Convention 4.1), let Q_k denote the set of all $\gamma \in EC(G - \mathcal{E}(P))$ such that $\gamma(\epsilon_{k-2}) = \gamma(\epsilon_k) = \gamma(\epsilon_{k+2})$.

It is understood (see Remark 4.3(A)) that for each $k \in \{0, 1, 2, 3, 4\}$ and each $\gamma \in Q_k$, the colors $\gamma(\epsilon_{k-1})$, $\gamma(\epsilon_k)$, and $\gamma(\epsilon_{k+1})$ are distinct.

By Remark 4.3(A)(B)(C) and the assumption that G is a snark,

$$EC(G - \mathcal{E}(P)) \;=\; \bigcup_{k \in \{0,1,2,3,4\}} Q_k \tag{4.2}$$

and (by a trivial argument) the sets Q_k are (pairwise) disjoint.

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Definition 2. In Context 0, define the mappings $M : EC(G-\mathcal{E}(P)) \to EC(G-\mathcal{E}(P))$ and $M^* : EC(G - \mathcal{E}(P)) \to EC(G - \mathcal{E}(P))$ as follows. The two definitions will be given together in three "steps":

Suppose $\beta \in EC(G - \mathcal{E}(P))$.

(i) Referring to (4.2) and the phrase right after it, let $\kappa \in \{0, 1, 2, 3, 4\}$ be the index such that $\beta \in Q_\kappa$.

(ii) Referring to Definition 1 and Remark 4.3(A), let the three distinct colors s_-, s_o, and s_+ (some permutation of the colors a, b, and c in (2.1)) be defined by $s_o := \beta(\epsilon_{\kappa-2}) = \beta(\epsilon_\kappa) = \beta(\epsilon_{\kappa+2})$ and $s_- := \beta(\epsilon_{\kappa-1})$ and $s_+ := \beta(\epsilon_{\kappa+1})$.

(iii)(a) Let $M\beta$ denote the element of $EC(G - \mathcal{E}(P))$ that one obtains from β by interchanging the colors s_o and s_- along the $s_o s_-$-Kempe chain containing the edge $\epsilon_{\kappa-1}$.

(iii)(b) Let $M^*\beta$ denote the element of $EC(G - \mathcal{E}(P))$ that one obtains from β by interchanging the colors s_o and s_+ along the $s_o s_+$-Kempe chain containing the edge $\epsilon_{\kappa+1}$.

Lemma 3. In Context 0, suppose $\gamma \in EC(G - \mathcal{E}(P))$. Referring to (4.2) and the phrase after it, let $k \in \{0, 1, 2, 3, 4\}$ be the index such that $\gamma \in Q_k$. Then (recall Convention 4.1) the following four statements hold:

(i) $M\gamma \in Q_{k+2}$;

(ii) $M^*\gamma \in Q_{k-2}$;

(iii) $M^* M\gamma = \gamma$; and

(iv) $MM^*\gamma = \gamma$.

Proof. The proofs of (ii) and (iv) are respectively exactly analogous to (are "mirror images" of) the proofs of (i) and (iii). It will suffice to give the argument for (i) and (iii). Here the arguments for (i) and (iii) will be given together. The remaining paragraphs in this proof will be labeled (P1), (P2), etc.

(P1) As in the hypothesis, suppose $\gamma \in EC(G - \mathcal{E}(P))$. Let $k \in \{0, 1, 2, 3, 4\}$ be the index such that $\gamma \in Q_k$. Let the (distinct) colors x, y, and z be defined by

$$x := \gamma(\epsilon_{k-2}) = \gamma(\epsilon_k) = \gamma(\epsilon_{k+2}) \quad \text{and} \quad y := \gamma(\epsilon_{k-1}) \quad \text{and} \quad z := \gamma(\epsilon_{k+1}). \qquad (4.3)$$

(P2) Referring to (4.3), let K be the xy-Kempe chain (for the coloring γ) containing the edge ϵ_{k-1}. By Definition 2, the edge-3-coloring $M\gamma$ of $G - \mathcal{E}(P)$ is obtained from the coloring γ by the interchanging of the colors x and y along that Kempe chain K.

(P3) Recall that in the graph $G - \mathcal{E}(P)$, the vertex v_{k-1} is univalent and is connected to the edge ϵ_{k-1}. It follows (recall Remark (d) in Definition 2.11) that K is a path (i.e. not a cycle) with two endpoints, one of which is v_{k-1} and the other is some other univalent vertex. Since the only univalent vertices in $G - \mathcal{E}(P)$ are the five vertices v_i, $i \in \{0, 1, 2, 3, 4\}$, the other endpoint of K must be one of those vertices v_i other than v_{k-1}. That other endpoint cannot be v_{k+1}, because the edge connected to it, namely ϵ_{k+1}, is colored z (not x or y) by γ — see (4.3). Hence that other endpoint must be either v_k, v_{k-2}, or v_{k+2}.

(P4) If the other endpoint of K (besides v_{k-1}) were v_{k+2}, then the mapping $M\gamma$ would assign the color x to the edges ϵ_k, ϵ_{k-1}, and ϵ_{k-2}, the color y to ϵ_{k+2}, and the color z to ϵ_{k+1}, and by Remark 4.3(B), the mapping $M\gamma$ would extend to an edge-3-coloring of the original graph G, contradicting the assumption that G is a snark.

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(P5) If the other endpoint of K (besides v_{k-1}) were v_k, then the mapping $M\gamma$ would assign the color x to the edges ϵ_{k-1}, ϵ_{k-2}, and $\epsilon_{k+2}(=\epsilon_{k-3})$, the color y to ϵ_k, and the color z to ϵ_{k+1}, and in this case too by Remark 4.3(B), the mapping $M\gamma$ would extend to an edge-3-coloring of the original graph G, contradicting the assumption that G is a snark.

(P6) Consequently, the other endpoint of K (besides v_{k-1}) has to be v_{k-2}.

(P7) The coloring $M\gamma$ assigns the following colors to the edges ϵ_i, $i \in \{0, 1, 2, 3, 4\}$: $(M\gamma)(\epsilon_{k-1}) = (M\gamma)(\epsilon_k) = (M\gamma)(\epsilon_{k+2}) = x$, $(M\gamma)(\epsilon_{k-2}) = y$, and $(M\gamma)(\epsilon_{k+1}) = z$. Using Convention 4.1, let us display this information in a slightly different way:

$$x = (M\gamma)(\epsilon_k) = (M\gamma)(\epsilon_{k+2}) = (M\gamma)(\epsilon_{k+4}) \quad \text{and}$$
$$z = (M\gamma)(\epsilon_{k+1}) \quad \text{and} \quad y = (M\gamma)(\epsilon_{k+3}). \tag{4.4}$$

By (4.4) and Definition 1, $M\gamma \in Q_{k+2}$. Thus statement (i) in Lemma 3 holds.

(P8) Our remaining task is to prove statement (iii) in Lemma 3. By (4.4) and Definition 2, since $M\gamma \in Q_{k+2}$, the edge-3-coloring $M^*M\gamma$ is obtained from $M\gamma$ by the interchanging of the colors x and y along the xy-Kempe chain K_1 (for the coloring $M\gamma$) containing the edge $\epsilon_{(k+2)+1} = \epsilon_{k+3} = \epsilon_{k-2}$. (Here the coloring β, the index κ, and the colors s_o and s_+ in Definition 2 are the coloring $M\gamma$, the index $k+2$, and the colors x and y in (4.4).)

(P9) For comparison to the Kempe chain K_1 (for $M\gamma$) in paragraph (P8), note that the xy-Kempe chain K (for the original coloring γ) in paragraph (P2) has the following two properties: (i) K contains the edge edge ϵ_{k-2}; and (ii) K is an xy-Kempe chain for the coloring $M\gamma$. Property (i) holds by paragraph (P6); and property (ii) holds by paragraph (P2) and Remark (e)(i) in Definition 2.11, By paragraph (P8), K_1 also satisfies (i) and (ii). Hence K_1 is identical to K, by the first sentence after Definition 2.11(b). Thus by paragraph (P8), the coloring $M^*M\gamma$ is obtained from $M\gamma$ by the interchanging of the colors x and y along K. Hence by paragraph (P2) and Remark (e)(ii) in Definition 2.11, statement (iii) in Lemma 3 holds. This completes the proof of Lemma 3.

Lemma 4. *Refer to Definition 1. In Context 0, for each $k \in \{0, 1, 2, 3, 4\}$, $\mathrm{card}\,Q_k = \mathrm{card}\,Q_{k+2}$.*

Proof. Suppose $k \in \{0, 1, 2, 3, 4\}$. Refer to Definition 2. Let us restrict the mapping M in Definition 2 to the domain Q_k, and let us also restrict the mapping M^* in Definition 2 to the domain Q_{k+2}. Under these restrictions, by Lemma 3(i)(ii), we thereby have that M maps Q_k into Q_{k+2}, and M^* maps Q_{k+2} into Q_k. By Lemma 3(iii) and the usual trivial argument, the (restricted) mapping M is one-to-one (as a mapping of Q_k into Q_{k+2}). By Lemma 3(iv) and the usual trivial argument, the (restricted) mapping M is also onto (as a mapping of Q_k into Q_{k+2}). Hence the (restricted) mapping M gives a one-to-one correspondence between the sets Q_k and Q_{k+2}. Lemma 4 follows.

Step 5. In Context 0, define the nonnegative integer $J := \mathrm{card}\,Q_0$. Then by four applications of Lemma 4,

$$J = \mathrm{card}\,Q_0 = \mathrm{card}\,Q_2 = \mathrm{card}\,Q_4 = \mathrm{card}\,Q_1 = \mathrm{card}\,Q_3. \tag{4.5}$$

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Hence by eq. (4.2) and the phrase right after it,

$$\operatorname{card} EC(G - \mathcal{E}(P)) \;=\; 5J. \tag{4.6}$$

Step 6. In Context 0, suppose e is any edge of P, and let $k \in \{0,1,2,3,4\}$ denote the index such that $e = (v_{k-2}, v_{k+2})$ (recall Convention 4.1).

If μ is any edge-3-coloring of the cubic graph G_e, then trivially there is a (unique) edge-3-coloring γ of $G - \mathcal{E}(P)$ that meets the following conditions (referring to vertices w_i in Context 0):

$$\gamma(\epsilon_{k-2}) = \mu((w_{k-2}, v_{k-1})), \quad \gamma(\epsilon_{k+2}) = \mu((w_{k+2}, v_{k+1})), \quad \text{and}$$
$$\gamma(\varepsilon) = \mu(\varepsilon) \text{ for every edge } \varepsilon \text{ of } G - \mathcal{E}(P) \text{ other than } \epsilon_{k-2} \text{ and } \epsilon_{k+2}. \tag{4.7}$$

Lemma 7. *In Context 0, suppose e is any edge of P, and let $k \in \{0,1,2,3,4\}$ denote the index such that $e = (v_{k-2}, v_{k+2})$ (recall Convention 4.1).*

(i) For any $\gamma \in Q_{k-2} \cup Q_k \cup Q_{k+2}$, there exists a unique edge-3-coloring μ of the cubic graph G_e such that (4.7) holds.

(ii) For any $\gamma \in Q_{k-1} \cup Q_{k+1}$, there does not exist an edge-3-coloring μ of G_e such that (4.7) holds.

Proof. Let us prove (ii) first. Consider first the case where $\gamma \in Q_{k-1}$. Suppose there were to exist an edge-3-coloring μ of G_e such that (4.7) holds. Then one would have $\gamma(\epsilon_{k-1}) = \gamma(\epsilon_{k+1}) = \gamma(\epsilon_{k+2})$. To the second and third terms there, one can apply (4.7), and one obtains (again recall the vertices w_i in Context 0)

$$\mu(\epsilon_{k+1}) = \gamma(\epsilon_{k+1}) = \gamma(\epsilon_{k+2}) = \mu((w_{k+2}, v_{k+1})). \tag{4.8}$$

However, since the vertex v_{k+1} is an endpoint of both of the edges (w_{k+2}, v_{k+1}) and ϵ_{k+1} (in G_e), one must have $\mu(\epsilon_{k+1}) \neq \mu((w_{k+2}, v_{k+1}))$, which contradicts (4.8). Thus (if $\gamma \in Q_{k-1}$) there cannot exist an edge-3-coloring μ of G_e such that (4.7) holds.

By an analogous ("mirror image") argument, one has that if $\gamma \in Q_{k+1}$, there cannot exist an edge-3-coloring μ of G_e such that (4.7) holds. That completes the proof of statement (ii) in Lemma 4.7.

Proof of statement (i). For a given edge-3-coloring γ of $G - \mathcal{E}(P)$ such that $\gamma \in Q_{k-2} \cup Q_k \cup Q_{k+2}$, in order to "extend" it to an edge-3-coloring of G_e — more precisely, in order to define an edge-3-coloring μ of G_e that satisfies (4.7) — one would need to assign colors $\mu(.)$ to the two remaining edges (v_{k-1}, v_k) and (v_k, v_{k+1}) (of G_e), in such as way as to avoid giving the same color to two adjacent edges.

Consider first the case where $\gamma \in Q_k$. Let the (three distinct) colors x, y, and z be as in (4.3) (in the proof of Lemma 3). There exists a unique edge-3-coloring of G_e such that (4.7) holds. It is obtained by assigning the following colors to the two remaining edges: $\mu((v_{k-1}, v_k)) := z$ and $\mu((v_k, v_{k+1})) := y$.

Next consider the case where $\gamma \in Q_{k-2}$. Let the (three distinct) colors x, y, and z be defined by

$$x := \gamma(\epsilon_{k+1}) = \gamma(\epsilon_{k-2}) = \gamma(\epsilon_k) \quad \text{and} \quad y := \gamma(\epsilon_{k+2}) \quad \text{and} \quad z := \gamma(\epsilon_{k-1}).$$

This is simply a version of (4.3) with -2 added to each index. In this case, there is a unique edge-3-coloring μ of G_e that satisfies (4.7). It is obtained by assigning the following colors to the two remaining edges: $\mu((v_{k-1}, v_k)) := y$ and $\mu((v_k, v_{k+1})) := z$.

For the remaining case $\gamma \in Q_{k+2}$ the argument is exactly analogous to (is a "mirror image" of) the argument for the case $\gamma \in Q_{k-2}$. That completes the proof of statement (i), and of Lemma 7.

Lemma 8. *In Context 0, for every edge e of the pentagon P, one has that*

$$\operatorname{card} EC(G_e) \ = \ 3J. \tag{4.9}$$

Proof. By Step 6 and Lemma 7, eq. (4.7) gives a one-to-one correspondence between the set of all edge-3-colorings μ of G_e and the set of all $\gamma \in Q_{k-2} \cup Q_k \cup Q_{k+2}$. Hence by (4.5) (and the phrase right after (4.2)), eq. (4.9) holds.

Step 9. In this step, it will be convenient to slightly abbreviate the earlier notation $\{\delta \in ED(G - \mathcal{E}(P)) : \epsilon_{k-2} \sim \epsilon_k \sim \epsilon_{k+2}\}$ to simply $\{ED(G - \mathcal{E}(P)) : \epsilon_{k-2} \sim \epsilon_k \sim \epsilon_{k+2}\}$.

Refer again to Context 0. By eq. (2.4) in Remark 2.6, one has that the integer $\operatorname{card} EC(G - \mathcal{E}(P))$ is a multiple of 6. It now follows from (4.6) that the (nonnegative) integer J must be a multiple of 6. Define the nonnegative integer L by $L := J/6$. Then by (4.6) and (4.9), together with (again) eq. (2.4),

$$\operatorname{card} ED(G - \mathcal{E}(P)) \ = \ 5L; \tag{4.10}$$

and

$$\operatorname{card} ED(G_e) = 3L \quad \text{for each edge } e \text{ of the pentagon } P. \tag{4.11}$$

Also, for any given $k \in \{0, 1, 2, 3, 4\}$, any $\delta \in \{ED(G - \mathcal{E}(P)) : \epsilon_{k-2} \sim \epsilon_k \sim \epsilon_{k+2}\}$ gives rise to exactly 6 edge-3-colorings $\gamma \in Q_k$ (with the edges in any of the three classes in the decomposition δ being given the same color), since there are exactly 6 permutations of the three colors a, b, and c. Of course any such coloring γ arises from exactly one such decomposition δ. Hence for each $k \in \{0, 1, 2, 3, 4\}$, $\operatorname{card} Q_k = 6 \cdot \operatorname{card} \{ED(G - \mathcal{E}(P)) : \epsilon_{k-2} \sim \epsilon_k \sim \epsilon_{k+2}\}$. Hence by (4.5),

$$\forall k \in \{0, 1, 2, 3, 4\}, \quad \operatorname{card} \{ED(G - \mathcal{E}(P)) : \epsilon_{k-2} \sim \epsilon_k \sim \epsilon_{k+2}\} = L. \tag{4.12}$$

Now by (3.7) and (4.11),

$$\psi(G, e) = L \quad \text{for each edge } e \text{ of the pentagon } P. \tag{4.13}$$

One now obtains sub-statement (i) in statement (B) (in Theorem 4.5) by substituting (4.13) into (4.10), and one obtains sub-statement (ii) in statement (B) by substituting (4.13) into (4.12). That, together with (4.13) itself, completes the proof of statement (B) (and of statement (A) for one pentagon) in Theorem 4.5.

Proof of statement (A). Recall that in the case where $H = P$ itself for some pentagon P in G, from (4.13) in the proof above (for statement (B) and for statement

(A) for this particular pentagon P), one already has established that the numbers $\psi(G, e)$, $e \in P$ are equal. This special case will be tacitly used below.

Now suppose instead that H is a connected union of two or more pentagons in G. Suppose e_1 and e_2 are any two distinct edges of H. It suffices to prove that

$$\psi(G, e_1) = \psi(G, e_2). \tag{4.14}$$

Since H is connected, there is a finite sequence of edges $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n$ in H such that $\varepsilon_0 = e_1$, $\varepsilon_n = e_2$, and for each $i \in \{0, 1, \ldots, n-1\}$, the edges ε_i and ε_{i+1} are adjacent. If one can show that $\psi(G, \varepsilon_i) = \psi(G, \varepsilon_{i+1})$ for each $i \in \{0, 1, \ldots, n-1\}$, then (4.14) will follow. Hence, it suffices to prove (4.14) for the case where e_1 and e_2 themselves are adjacent.

Suppose e_1 and e_2 are adjacent. If they belong to the same pentagon in H, then from the comments above (involving (4.13)), $\psi(G, e_1) = \psi(G, e_2)$ and we are done. Therefore, suppose instead that e_1 and e_2 do *not* belong to the same pentagon in H.

Let v denote the common endpoint vertex of the two edges e_1 and e_2. Let e_3 be the third edge connected to v. Now (by hypothesis) e_1 belongs to some pentagon P_1 in H. This pentagon P_1 does not contain the edge e_2. It follows that P_1 is forced to contain the edge e_3 (since every vertex in P_1, including v, is connected to two edges in P_1). Hence $\psi(G, e_1) = \psi(G, e_3)$. Similarly, e_2 belongs to some pentagon P_2 in H, P_2 does not contain e_1, hence P_2 must contain e_3, and hence $\psi(G, e_2) = \psi(G, e_3)$. Eq. (4.14) now follows. That completes the proof of statement (A), and of Theorem 4.5.   ////

Definition 4.6. Refer to Theorem 4.5(A). If G is a snark and H is a connected union of pentagons in G (or H is simply a single pentagon in G), then define the nonnegative integer

$$\psi(G, H) := \psi(G, e) \tag{4.15}$$

where e is any edge of H.

The following well known procedure for combining two snarks, each having a pentagon, to create a "bigger" snark, was first used by Isaacs [Is, pp. 234-236] to create his "double star" snark from two disjoint copies of the flower snark snark J_5 (see Remark 2.14, and note that J_5 has a lone pentagon). This procedure also plays a role in the "decomposition" of a "big" snark with a (nontrivial) 5-edge cut set into two "smaller" snarks; see e.g. [CCW, Theorem 2].

Context 4.7. Suppose G' is a snark with a pentagon P' with vertices u_k and edges (u_k, u_{k+1}), $k \in \mathbf{Z}_5$ (recall Convention 4.1). Let $G'_0 := G' - \{u_k, k \in \mathbf{Z}_5\}$ (the graph that one derives from G' by deleting the five vertices u_k and all ten edges connected to them, including the five edges in the pentagon P'). For each $k \in \mathbf{Z}_5$, let t_k denote the vertex of G'_0 such that (t_k, u_k) is an edge of G'. Since G' is a snark (and hence has no 3-cycles or 4-cycles), these vertices t_k, $k \in \mathbf{Z}_5$ are distinct.

Suppose G^* is a snark that is disjoint from G' and has a pentagon P^* with vertices v_k and edges (v_k, v_{k+1}), $k \in \mathbf{Z}_5$. Let $G^*_0 := G^* - \{v_k, k \in \mathbf{Z}_5\}$. For each $k \in \mathbf{Z}_5$, let w_k

denote the vertex of G_0^* such that (v_k, w_k) is an edge of G^*. These vertices w_k, $k \in \mathbf{Z}_5$ are distinct.

Let G denote the simple cubic graph that consists of G_0', G_0^*, and the five new edges (t_k, w_{2k}), $k \in \mathbf{Z}_5$ (again recall Convention 4.1).

As is well known — e.g. from Isaacs [Is] in connection with his "double star" snark alluded to above, right before Context 4.7 — the graph G in Context 4.7 is a snark. Here let us quickly review the well known proof (from [Is, Theorem 4.2.1]) that the graph G cannot be edge-3-colored. (To verify that G is at least 4-edge-connected and has girth at least 5, requires separate arguments, which we shall not go into here.)

Suppose (to seek a contradiction) γ were an edge-3-coloring of G. Then by Remark 2.12(d) (see Lemma 2.13 and adapt the Remark after it), one has that (i) for (exactly) three indices $q, r, s \in \mathbf{Z}_5$, $\gamma((t_q, w_{2q})) = \gamma((t_r, w_{2r})) = \gamma((t_s, w_{2s}))$, and (ii) for the other two indices i and j in $\mathbf{Z}_5$, the colors $\gamma((t_q, w_{2q}))$, $\gamma((t_i, w_{2i}))$, and $\gamma((t_j, w_{2j}))$ are distinct. If the three indices q, r, and s are (in some order) consecutive in $\mathbf{Z}_5$, then γ would induce an edge-3-coloring of G' (for each $k \in \mathbf{Z}_5$, the color $\gamma((t_k, w_{2k}))$ is given to the edge (t_k, u_k) in G', and then one applies Remark 4.3(B)); but that would contradict the assumption that G' is a snark. If instead the three indices q, r, and s are not all consecutive in $\mathbf{Z}_5$ (in any order), then by Remark 4.2(C), the indices $2q$, $2r$, and $2s$ *are* (in some order) consecutive in $\mathbf{Z}_5$, and γ would analogously induce an edge-3-coloring of G^*, contradicting the assumption that G^* is a snark. Thus an edge-3-coloring γ of G cannot exist.

Theorem 4.8. *In Context 4.7, the following two statements hold (see Definition 4.6):*
(A) For any edge e of G_0^*, e is an edge of G and

$$\psi(G, e) = \psi(G^*, e) \cdot \psi(G', P') . \tag{4.16}$$

(B) For any edge e of G_0', e is an edge of G and

$$\psi(G, e) = \psi(G', e) \cdot \psi(G^*, P^*) . \tag{4.17}$$

This theorem, and its proof given below, are due to the author [Br2, Theorem 3.2 and the sentence after it]. The proof was given there somewhat tersely. It will be repeated here in more generous detail.

Proof. It will suffice to give the argument for statement (A). The proof for statement (B) is exactly analogous, and will not be given explicitly here.

Proof of statement (A). As in the hypothesis of statement (A), suppose e is an edge of G_0^*. Then trivially from Context 4.7, e is an edge of G. For notational convenience, we shall carry out the proof here for the case where e is not connected to any of the vertices w_k, $k \in \mathbf{Z}_5$. (If instead e were connected to one or two of those vertices w_k, the proof would be essentially the same, with only minor notational changes.)

Suppose γ is a coloring of G_e. Then γ induces the following coloring γ' of the quasi-cubic graph $G' - \mathcal{E}(P')$:

$$\gamma'(d) := \gamma(d) \quad \text{for } d \in G'_0, \quad \text{and}$$
$$\gamma'((t_k, u_k)) := \gamma((t_k, w_{2k})) \quad \text{for } k \in \mathbf{Z}_5. \tag{4.18}$$

By Remark 2.12(d), applied to (say) $G' - \mathcal{E}(P')$, the colors $\gamma((t_k, w_{2k}))$, $k \in \mathbf{Z_5}$ are the same for three indices $k = q, r, s$ and distinct for the other two (i.e. with the other two colors each appearing on exactly one of those other two edges). By Remark 4.3(A)(B) and the assumption that G' is a snark, those three indices q, r, s cannot all be (in any order) consecutive in $\mathbf{Z}_5$ (mod 5). Hence by Remark 4.2(C), the indices $2q, 2r, 2s$ are (in some order) all consecutive in $\mathbf{Z}_5$. Hence by Remark 4.3(B), γ induces a unique coloring γ^* of G_e^* as follows:

$$\gamma^*(d) := \gamma(d) \quad \text{for } d \in G_e^* - \{v_k, k \in \mathbf{Z_5}\}, \quad \text{and}$$
$$\gamma^*((v_{2k}, w_{2k})) := \gamma((t_k, w_{2k})) \quad \text{for } k \in \mathbf{Z}_5. \tag{4.19}$$

(The colors $\gamma^*(d)$ for the edges d of P^* are not specified here; they will be uniquely determined by the colors $\gamma^*((v_{2k}, w_{2k}))$, $k \in \mathbf{Z}_5$.) Thus the coloring γ of G_e induces an ordered pair (γ', γ^*) such that (see (4.18) and (4.19))

$$\gamma' \text{ is a coloring of } G' - \mathcal{E}(P),$$
$$\gamma^* \text{ is a coloring of } G_e^*, \quad \text{and}$$
$$\gamma'((t_k, u_k)) = \gamma^*((v_{2k}, w_{2k})) \quad \text{for } k \in \mathbf{Z}_5. \tag{4.20}$$

Conversely, an ordered pair (γ', γ^*) as in (4.20) induces a (unique) coloring γ of G_e via (4.18) and (4.19). Thereby one has a one-to-one correspondence between colorings γ of G_e and ordered pairs (γ', γ^*) satisfying (4.20).

Now suppose γ^* is a coloring of G_e^*. Then by Remark 4.3(A)(B)(C), there exists a permutation x, y, z of the colors (a, b, and c) and an element ℓ of $\mathbf{Z}^5$ such that

$$\gamma^*((v_k, w_k)) = \begin{cases} x & \text{for } k \in \{\ell - 1, \ell, \ell + 1\} \\ y & \text{for } k = \ell - 2 \\ z & \text{for } k = \ell + 2. \end{cases} \tag{4.21}$$

Let $j \in \mathbf{Z}_5$ be defined by $j := 3\ell$ (mod 5) (and hence $2j = 6\ell = \ell$ (mod 5)). If γ' is a coloring of the quasi-cubic graph $G' - \mathcal{E}(P)$, then (the last line of) (4.20) holds if and only if

$$\gamma'((t_k, u_k)) = \begin{cases} x & \text{for } k \in \{j - 2, j, j + 2\} \\ y & \text{for } k = j - 1 \\ z & \text{for } k = j + 1. \end{cases} \tag{4.22}$$

By (4.15) and Theorem 4.5(B)(ii) (and Remark 4.3(D)), there are exactly $\psi(G', P')$ 3-edge-decompositions of $G' - \mathcal{E}(P')$ such that the edges (t_k, u_k), $k \in \{j - 2, j, j + 2\}$ are in the same class (in the decomposition) and the edges (t_k, u_k), $k \in \{j - 1, j + 1\}$ are in the

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other two classes respectively. Hence (similarly to Remark 2.6) there are exactly $\psi(G', P')$ colorings γ' of $G' - \mathcal{E}(P')$ such that (4.22) holds, equivalently such that the ordered pair (γ', γ^*) is as in (4.20). We have shown that this holds for an arbitrary coloring γ^* of G_e^*.

Recall from Definition 3.4 that there are exactly $18 \cdot \psi(G^*, e)$ colorings γ^* of G_e. Hence by the preceding paragraph, there are exactly $18 \cdot \psi(G^*, e) \cdot \psi(G', P')$ ordered pairs (γ', γ^*) satisfying (4.20), and hence that is the number of colorings of G_e. Hence by Definition (3.4), eq. (4.16) holds. That completes the proof of statement (A), and of Theorem 4.8.   ////

5. An example of $\psi(G, e)$ for a superposition

Isaacs [Is] presented some methods for creating arbitrarily large snarks. One of those methods involved a particular procedure (a "four-edge connection", which Isaacs called a "dot product") for "combining" two disjoint "smaller" snarks to form a "bigger" snark. (The formal definition of "dot product" can be found in [Is] or [Ga], and the procedure will be described just informally in comments after Context 5.2 below. See also the comments in Section 7.2(B).) In general, in the "dot product", the roles of the two "smaller" snarks are not "symmetric" to each other. However, in a certain class of special cases, the roles of the two "smaller" snarks (when looked at in the right way) are in fact "symmetric" to each other. We shall allude to that type of special case here (as in [Br2]) as a "symmetric dot product".

If G_1 and G_2 are two disjoint snarks, and G is a snark obtained from G_1 and G_2 via a "symmetric dot product", then for every edge e of G, there is a "natural" choice of edges e_1 of G_1 and e_2 of G_2 such that $\psi(G, e)$ is either equal to $2 \cdot \psi(G_1, e_1) \cdot \psi(G_2, e_2)$ or equal to $3 \cdot \psi(G_1, e_1) \cdot \psi(G_2, e_2)$.

That "fact" was given a precise formulation in [Br2, Theorem 2.2 and subsequent sentence]. We shall not elaborate further on it here. That "fact" (the precise formulation of it) was applied in an induction argument (using the Petersen graph and Theorem 3.5 as the starting point as well as in the induction step) to prove the following theorem ([Br2, Theorem 1.4]):

Theorem 5.1. *Suppose j and k are each a nonnegative integer. Then there exists a snark G and an edge e of G such that $\psi(G, e) = 2^j \cdot 3^k$.*

In that paper [Br2], the following open problem was implicitly posed: *For precisely what positive integers n do there exist a snark G and an edge e of G such that $\psi(G, e) = n$?*

The purpose of this paper here is to promote research on this problem and on other closely related problems, such as those posed in Section 7.

For the problem here, to make further progress beyond Theorem 5.1, one will need to apply much broader and more flexible techniques for creating "big" snarks from "small" ones than just the "symmetric dot product". Such a broad, flexible technique was devised by Kochol [Ko1], who referred to it under the name "superpositions". That technique appears to have much promise for this problem and related ones.

Kochol's [Ko1] technique involves starting with a snark, and "replacing" one or more of its edges by snarks in certain ways (together with "replacing" the end-point vertices of those edges by, say, quite general appropriate quasi-cubic graphs — not necessarily snarks), to create a "bigger" snark. For a particularly fascinating application of Kochol's technique, the reader is referred to the paper [Ko1] itself (starting with Theorem 1 there and its proof), where snarks with arbitrarily large girth are constructed. If the notations in that paper seem a little bewildering at first, the diagrams in that paper tell the key features of that story very well — in light of alert, thoroughly pervasive use of both Lemma 2.13 (the Parity Lemma) above and the properties of the Petersen graph.

Isaacs' "dot product", alluded to above, can be regarded as the simplest example of a Kochol superposition. (A little more on that below.) The rest of this section will be devoted to an illustration of another simple Kochol superposition, in connection with the numbers $\psi(G, e)$ for a given snark G and edge e of G. This particular superposition will provide a good simple illustration of, and motivation for, the use of the group arithmetic on $\mathbf{Z}_2 \times \mathbf{Z}_2$ (as in Definition 2.1) in the study of problems involving the numbers $\psi(G, e)$.

Context 5.2. Suppose G' is a snark, and $E = (U, V)$ is an edge of G'. Let $G'_0 = G' - \{U, V\}$. Suppose T_{-2}, T_2 are the vertices of G'_0 such that (T_i, U), $i \in \{-2, 2\}$ are edges of G'. Suppose W_{-2}, W_2 are the vertices of G'_0 such that (V, W_i), $i \in \{-2, 2\}$ are edges of G'.

Suppose G^* is a snark that is disjoint from G'. Suppose u and v are (distinct) vertices of G^* that are not adjacent to each other in G^*. Suppose u_i, $i \in \{-1, 0, 1\}$ are the vertices of G^* such that (u, u_i) is an edge of G^*. Suppose v_i, $i \in \{-1, 0, 1\}$ are the vertices of G^* such that (v_i, v) is an edge of G^*. Let $G_0^* := G^* - \{u, v\}$.

Let G denote the simple cubic graph that consists of G'_0, G_0^*, six new vertices T_i and W_i, $i \in \{-1, 0, 1\}$, and fourteen new edges (T_i, T_{i+1}) and (W_i, W_{i+1}), $i \in \{-2, -1, 0, 1\}$ and (T_i, u_i) and (v_i, W_i), $i \in \{-1, 0, 1\}$.

(One of the vertices u_i may be equal to one of the vertices v_j. Any further equalities of those vertices is prohibited by the requirement that the snark G^* have girth at least 5 — recall Definition 2.9.)

Context 5.2 gives one way (certainly not the only one) to "replace" the edge E in the snark G' by the snark G^* — that is, within the snark G', to "superimpose" the snark G^* in place of the edge E. As part of the known information pertaining to Kochol's superpositions, it is well known that (under the assumption that G' and G^* are each a snark) the new graph G is a snark. As a review of the proof that G cannot be colored, one can simply carry out the proof of Theorem 5.3 below, but with G_e and G_e^* replaced by G and G^*; at (5.7) one would then obtain a coloring of the snark G^*, a contradiction. It takes extra arguments (which we shall not go into here) to verify for G the other properties in the definition of "snark" (Definition 2.9).

In the above superposition of the snark G^* in place of the edge E, each of two (non-adjacent) vertices of G^* were "split" into three "strands" that were then "hooked up to G' where a vertex of G' used to be".

In other possible superpositions of G^* in place of E, one might instead "split" each of two nonadjacent *edges* of G^* into two "strands" to be similarly reattached. In its simplest form, that is actually what is done in Isaacs' [Is] "dot product" alluded to above — producing a "4-edge connection" that "combines" two snarks to form a bigger snark. In a different but closely related context (colorable simple cubic graphs with orthogonal edges), essentially the same procedure was also formulated slightly earlier by Kászonyi [Ká1]; see Section 7.2(B) in Section 7.

As yet another obvious alternative, one might "split" one edge (of G^*) into two "strands" and one vertex (not connected to that edge) into three "strands".

In a superposition studied by McKinney [McK] (somewhat more complicated than that in Context 5.2), a pair of adjacent edges of a snark was "replaced" by a pair of disjoint Petersen graphs. That will be discussed briefly in Section 6 below. (That "double superposition" involved the "splitting" of both edges and vertices.)

Theorem 5.3 *In Context 5.2, suppose e is an edge of G_0^*. Then e is an edge of G, and*

$$\psi(G, e) = 2 \cdot \psi(G^*, e) \cdot \psi(G', E). \tag{5.1}$$

Proof. For convenience of notations, the proof will be carried out in the case where the edge e is not connected to any of the vertices u_i, v_i, $i \in \{-1, 0, 1\}$. (In the case where e is connected to one of those vertices, the proof is essentially the same, with only trivial changes in notations.) Refer to Definition 2.1. In the argument below, considerable use will be made of elementary properties of the group $\mathbf{Z}_2 \times \mathbf{Z}_2$ (whose three nonzero elements are the three colors).

Suppose γ is a coloring of G_e. By Lemma 2.13 (and the Remark after it),

$$\gamma((T_{-2}, T_{-1})) + \gamma((T_1, T_2)) + \sum_{k=-1}^{1} \gamma((T_k, u_k)) = 0; \tag{5.2}$$

$$\gamma((W_{-2}, W_{-1})) + \gamma((W_1, W_2)) + \sum_{k=-1}^{1} \gamma((v_k, W_k)) = 0; \tag{5.3}$$

and

$$\sum_{k=-1}^{1} \gamma((T_k, u_k)) + \sum_{k=-1}^{1} \gamma((v_k, W_k)) = 0. \tag{5.4}$$

Since each element of $\mathbf{Z}_2 \times \mathbf{Z}_2$ is its own inverse, it follows that for some $x \in \mathbf{Z}_2 \times \mathbf{Z}_2$,

$$\begin{aligned}
x &= \sum_{k=-1}^{1} \gamma((T_k, u_k)) = \sum_{k=-1}^{1} \gamma((v_k, W_k)) \\
&= \gamma((T_{-2}, T_{-1})) + \gamma((T_1, T_2)) = \gamma((W_{-2}, W_{-1})) + \gamma((W_1, W_2)). \tag{5.5}
\end{aligned}$$

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In (5.5), if $x \in \{a, b, c\}$ were to hold, then by (5.5) and Remark 2.2 (and the fact that in $\mathbf{Z}_2 \times \mathbf{Z}_2$, $x = y + z \Longleftrightarrow 0 = x + y + z$), one would obtain a coloring β of the snark G' (and hence a contradiction) by defining $\beta(d) := \gamma(d)$ for $d \in \mathcal{E}(G_0')$, together with

$$\beta((T_{-2}, U)) := \gamma((T_{-2}, T_{-1})),$$
$$\beta((T_2, U)) := \gamma((T_2, T_1)),$$
$$\beta((W_{-2}, V)) := \gamma((W_{-2}, W_{-1})),$$
$$\beta((W_2, V)) := \gamma((W_2, W_1)), \quad \text{and}$$
$$\beta(E) := x.$$

Hence $x = 0$ instead. Thus by (5.5),

$$\sum_{k=-1}^{1} \gamma((T_k, u_k)) \;=\; \sum_{k=-1}^{1} \gamma((v_k, W_k)) \;=\; 0. \tag{5.6}$$

Hence by Remark 2.2, γ induces a coloring γ^* of G_e^* defined by

$$\gamma^*(d) := \gamma(d) \quad \text{for } d \in \mathcal{E}(G_e^* - \{u, v\}),$$
$$\gamma^*((u, v_k)) := \gamma((T_k, u_k)) \quad \text{for } k \in \{-1, 0, 1\}, \quad \text{and}$$
$$\gamma^*((v_k, v)) := \gamma((v_k, W_k)) \quad \text{for } k \in \{-1, 0, 1\}. \tag{5.7}$$

Also, using (5.5) and the fact that $x = 0$, and then using a trivial extra coloring argument, one has that

$$\gamma((T_{-2}, T_{-1})) = \gamma((T_1, T_2)) = \gamma((T_0, u_0)), \quad \text{and}$$
$$\gamma((W_{-2}, W_{-1})) = \gamma((W_1, W_2)) = \gamma((v_0, W_0)). \tag{5.8}$$

Hence γ induces a coloring γ' of G_E' defined by

$$\gamma'(d) := \gamma(d) \quad \text{for } d \in \mathcal{E}(G_0'),$$
$$\gamma'((T_{-2}, T_2)) := \gamma((T_0, u_0)), \text{and}$$
$$\gamma'((W_{-2}, W_2)) := \gamma((v_0, W_0)). \tag{5.9}$$

Thus a given coloring γ of G_e induces an ordered pair (γ^*, γ') where (see (5.7) and (5.9))

$$\gamma^* \text{ is a coloring of } G_e^*,$$
$$\gamma' \text{ is a coloring of } G_E',$$
$$\gamma'((T_{-2}, T_2)) = \gamma^*((u, u_0)), \quad \text{and}$$
$$\gamma'((W_{-2}, W_2)) = \gamma^*((v_0, v)). \tag{5.10}$$

Conversely, if (γ^*, γ') is an ordered pair satisfying (5.10), then it induces a unique coloring γ of G_e via (5.7) and (5.9). (The colors $\gamma(d)$ for the remaining four edges $d =$

$(T_k, T_0), (W_k, W_0), k \in \{-1, 1\}$ will trivially be uniquely determined.) Thereby one obtains a one-to-one correspondence between colorings γ of G_e and ordered pairs (γ^*, γ') as in (5.10).

Now by Definition 3.4, the number of colorings of G_e^* is $18 \cdot \psi(G^*, e)$. For each coloring γ^* of G_e^*, by Theorem 3.3(C)(3), there are exactly $2 \cdot \psi(G', E)$ colorings γ' of G'_E such that the two equalities in (5.10) hold. Hence there are exactly $36 \cdot \psi(G^*, e), \psi(G', E)$ ordered pairs (γ^*, γ') as in (5.10), and hence exactly that many colorings of G_e. Hence by Definition 3.4, eq. (5.1) holds. That completes the proof of Theorem 5.3.   ////

6. Results of McKinney

The paper [Br2], giving Theorem 5.1 and its proof, was published in March 2006. A few months later, in the summer of 2006, in a Mathematics REU (Research Experience for Undergraduates) program at Indiana University (organized by Professor Victor Goodman of the Indiana University Mathematics Department), Scott A. McKinney, an undergraduate mathematics major at Cornell University, did some research on snarks under the direction of the author of this paper. The results of that research were written up in a paper by McKinney [McK], as part of a collection of papers by the students in that REU program. Those results were of a spirit similar to Theorem 5.1 and Theorem 5.3, and included an extension of Theorem 5.1 itself. The purpose of this section here is to summarize those results of McKinney [McK]. We shall start by stating the two main results of that paper:

Theorem 6.1 (McKinney [McK, Theorem 5.5]). *Suppose j and k are each a non-negative integer. Then there exists a cyclically 5-edge-connected snark G and an edge e of G such that $\psi(G, e) = 5^j \cdot 7^k$.*

Theorem 6.2 (McKinney [McK, Corollary 5.6]). *Suppose j, k, ℓ, and m are each a nonnegative integer. Then there exists a snark G and an edge e of G such that $\psi(G, e) = 2^j \cdot 3^k \cdot 5^\ell \cdot 7^m$.*

(In Theorem 6.2, the snark G need not be cyclically 5-edge-connected, though by Definition 2.9 it must of course be at least cyclically 4-edge-connected.)

Obviously Theorem 6.2 generalizes Theorem 5.1. McKinney [McK] obtained Theorem 6.1 first, and then mimicked the induction argument in [Br2] for Theorem 5.1 in order to derive Theorem 6.2 as a corollary of Theorem 6.1. Theorem 6.1 itself was obtained in [McK] by an induction argument that started with the Petersen graph and Theorem 3.5 and then (in the induction step) involved a particular choice of Kochol [Ko1] "superposition". In Remark 6.3 below, we shall give just a brief description of that whole process.

Remark 6.3. (a) The context studied by McKinney [McK] was a follows: It started with an arbitrary snark G_0. Two adjacent edges of that snark were "replaced" together by two (disjoint) Petersen graphs in a certain way, to create a new, "bigger" snark G. In

that "double superposition" process (as we shall call it here for convenience), in each of those two Petersen graphs, one edge was "split" into two "strands" and one vertex (not connected to that edge or even adjacent to an end-point vertex of it) was "split" into three "strands"; and the resulting ten "strands" were then "tied up" to each other and to (what was left of) G_0 in a particular way, creating a "five edge connection" between (what was left of) G_0 and the union of (what was left of) the two Petersen graphs. Although the details were quite different, the general spirit was somewhat like that of Context 5.2 (where one edge of a snark was "replaced" by another snark, in a "superposition" process in which two non-adjacent vertices of that "other" snark were each "split" into three "strands").

(b) McKinney [McK, Theorem 5.3] showed that for any given edge e of the original snark G_0 that was not involved in the "double superposition by two Petersen graphs" (and hence e was also an edge of the new snark G), one has that $\psi(G, e) = 5 \cdot \psi(G_0, e)$.

(c) McKinney also studied the edge $\mathbf{e}$ of G_0 that was (different from and) adjacent to the two edges of G_0 that were "replaced" by Petersen graphs in the "double superposition". In that "double superposition" process, the edge $\mathbf{e}$ itself was removed, and in the new snark G (regardless of the original choice of G_0) it had a "natural counterpart" — a new edge E. McKinney [McK, Theorem 5.1] showed that $\psi(G, E) = 7 \cdot \psi(G_0, \mathbf{e})$.

(d) In proving both of the results described in paragraphs (b) and (c) above, McKinney [McK] employed arguments that were, while quite different in their details, somewhat of the general spirit of the proof of Theorem 5.3. As mentioned above, McKinney [McK] then used induction, starting with the Petersen graph and Theorem 3.5 and then employing the results in both paragraphs (b) and (c) above in the induction step, to prove Theorem 6.1; and he then mimicked the induction argument in [Br2] in order to derive Theorem 6.2 as a corollary.

7. Some open problems

Section 7.1. Some open problems. Given below is a list of some open problems involving snarks and the numbers $\psi(G, e)$, and also involving edge-3-colorable cubic graphs with orthogonal edges. These problems are motivated by the papers of Kászonyi [Ká1, Ká2, Ká3] and are rooted primarily in the material in Section 3 of this survey paper. The first problem was implicitly posed by the author in [Br2] and was mentioned in Section 5 above.

Problem 1. For precisely what positive integers n do there exist a snark G and an edge e of G such that $\psi(G, e) = n$?

This problem can perhaps be approached via the following closely related one:

Problem 2. For what prime numbers p does the following (uncertain) "Hypothesis $S(p)$" hold?

Hypothesis $S(p)$: If n is a positive integer, G is a snark, e is an edge of G, and $\psi(G, e) = n$, then there exist a snark $\mathcal{G}$ and an edge E of $\mathcal{G}$ such that $\psi(\mathcal{G}, E) = n \cdot p$.

Now Hypothesis $S(p)$ was (implicitly) verified for $p = 2, 3$ by [Br2, Theorem 2.2] (one combines that result with Theorem 3.5, using the Petersen graph); and it was verified for $p = 5, 7$ by McKinney [McK, Theorems 5.3 and 5.1] (see Remark 6.3(b)(c) above). The next two problems are variations on Problem 1, and are motivated by Theorem 6.1. Perhaps they can be approached by corresponding variations on Problem 2.

Problem 3. For precisely what positive integers n do there exist a cyclically 5-edge-connected snark G and an edge e of G such that $\psi(G, e) = n$?

Problem 4. For precisely what positive integers n do there exist a cyclically 6-edge-connected snark G and an edge e of G such that $\psi(G, e) = n$?

Problem 4 is motivated partly by the "flower" snarks in Remark 2.14. The flower snarks J_n for $n \in \{7, 9, 11, \ldots\}$ (but not J_5) are known to be cyclically 6-edge-connected. The next question seems to be of obvious special interest:

Problem 5. What are the numbers $\psi(G, e)$ for the flower snarks G and their edges e?

For a given flower snark G, the edges fall into four equivalence classes in which two edges e_1 and e_2 are "equivalent" if there is an automorphism of G in which e_1 is mapped to e_2. Of course $\psi(G, e_1) = \psi(G, e_2)$ for any two such "equivalent" edges e_1 and e_2. (Recall the Remark after Definition 3.4.) This suggests that as a solution of Problem 5, there might be a "recursion formula" that involves as a parameter the subscript $n \in \{5, 7, 9, 11, \ldots\}$ for a given flower snark J_n (in the notations of Remark 2.14), and that also involves the four numbers $\psi(J_n, e)$ corresponding to the four "equivalence classes" of edges e of J_n.

The next two problems involve the material in Section 4.

Problem 6. Refer to Definition 4.6. For what positive integers n do there exist a snark G and a pentagon P in G such that $\psi(G, P) = n$?

The answer to this question is affirmative for $n = 1$ by the result of Kászonyi [Ká1, Ká2] for Petersen graphs given in Theorem 3.5. It is affirmative for powers of 2 by a simple observation in the context of the proof of [Br2, Theorem 2.2]. It is affirmative for powers of 5 by a simple observation in the context of an argument of McKinney [McK, Theorem 5.3] (see Remark 6.3(b) above). Combining arguments in those two references, one obtains that the answer is in fact affirmative for (at least) the numbers n of the form $2^j \cdot 5^\ell$, where j and ℓ are each a nonnegative integer.

Problem 7. In Theorem 4.8, five edges of G were omitted: the five "connecting" edges (t_k, w_{2k}), $k \in \mathbf{Z}_5$. If e is one of those five edges, what can one say about $\psi(G, e)$ in terms of $\psi(G', e')$ and $\psi(G^*, e^*)$ for appropriate edges e' and e^* of (respectively) G' and G^*?

From the solutions to Problems 5 and 7, or perhaps more easily from a direct argument, one might compute the numbers $\psi(G, e)$ for all of the edges e of the "double star" snark G alluded to right before Context 4.7.

The remaining problems below come directly (at least implicitly) from the work of Kászonyi [Ká1, Ká2].

Refer to Definition 3.1. In the proof of Theorem 3.5, for the Petersen graph $\mathcal{P}$ and an edge e of $\mathcal{P}$, the graph $\mathcal{P}_e$ was represented (as in [Ká1, Ká2]) as an "8-vertex wheel with four rim-to-rim spokes". In the notations used there (in the proof of Theorem 3.5) for that graph, the orthogonal edges resulting directly from the "removal" of the edge e from $\mathcal{P}$ were denoted f_0 and f_2. (Those were the edges corresponding to d_1 and d_2 in Notations 3.2.) However, by simple symmetry (simply "rotate the wheel 45 degrees"), that graph $\mathcal{P}_e$ has another pair of orthogonal edges: f_1 and f_3. This suggests the following problem:

Problem 8. If G is a snark, e is an edge of G, and G_e can be colored (i.e. $\psi(G, e) \geq 1$), does the cubic graph G_e have (at least) two pairs of orthogonal edges? Or are there instead examples where G_e has only one pair of orthogonal edges (the pair identified by Kászonyi [Ká2] in Theorem 3.3(B) — the edges d_1 and d_2 in Notations 3.2)? If the latter is the case, then under what extra assumptions on the snark G and the edge e of G does G_e have at least two pairs of orthogonal edges? (Just one pair?)

For the final question, Problem 9 below, a definition will be given first: Suppose G is a simple cubic graph which, say, is (at least) cyclically 4-edge-connected and has girth at least 5. (No assumption on whether or not G can be edge-3-colored.) Suppose e is an edge of G. Let d_1 and d_2 be the edges of G_e specified in Notations 3.2. Let us say that the edge e satisfies Condition $\mathcal{K}$ (for Kászonyi) if (i) the (simple cubic) graph G_e can be edge-3-colored, and (ii) the edges d_1 and d_2 are orthogonal (again see Definition 3.1).

Problem 9. Suppose G is a simple cubic graph which (say) is (at least) cyclically 4-edge-connected and has girth at least 5. If some edge of G satisfies Condition $\mathcal{K}$ (see the preceding paragraph above), does it follow that G is a snark?

This question is quite specific. If the answer is "no", then there are obvious variations on this question. For example, what if at least two edges of G satisfy Condition $\mathcal{K}$? If the answer is still "no", then (for example) what if all five edges of some pentagon (if one exists) in G satisfy Condition $\mathcal{K}$?

Section 7.2. Final Remarks. Here are some final comments on the papers of Kászonyi [Ká1, Ká2, Ká3] on which this survey paper is based.

(A) For a long time, those three papers of Kászonyi did not seem to be known much in the "snark community". (The author of this survey paper has not found any citations to those papers of Kászonyi in other published papers prior to their citations in the 2006 paper [Br2].)

(B) In Section 5 (its first paragraph and a couple of other places), the paper of Isaacs [Is] is cited for a "dot product" of two snarks — a particular "4-edge-connection" procedure for combining two "smaller" snarks to form a "bigger" one. In fact a few years earlier, Kászonyi [Ká1, pp. 86-87, Operation 3] had presented an exactly analogous "4-edge-connection" for combining two simple cubic graphs, each of them being edge-3-colorable with a pair of orthogonal edges (one of those graphs being $\mathcal{P}_e$ for an edge e of the Petersen graph $\mathcal{P}$), to form a "bigger" simple cubic graph which is edge-3-colorable with a pair of orthogonal edges.

(C) *Acknowledgement of priority.* Certain results and arguments of Kászonyi [Ká2, Ká3] — roughly (recall the first paragraph of Section 3), statements and proofs of Theorem 3.3(B), Theorem 3.3(C)(1)(2), and Theorem 3.7(B) — were independently rediscovered a few years later by the author [Br1, Theorem 1, Lemmas 1, 2, and 8, Corollary 2, and their proofs]. The priority for those results and arguments belongs to Kászonyi.

(D) The paper [Ká3] is somewhat cryptic. For a given edge-3-colorable simple cubic graph H with orthogonal edges, Kászonyi [Ká3] defined a "coloring graph", which will be referred to here as $\mathcal{H}$. The "vertices" of $\mathcal{H}$ correspond to edge-3-colorings of H with three given "colors" (say the elements a, b, and c from (2.1)). Two "vertices" of $\mathcal{H}$ are connected by an "edge" of $\mathcal{H}$ if the two corresponding edge-3-colorings of H differ from each other by just the interchanging of the two colors on one Kempe cycle. The observations made by Kászonyi [Ká3, p. 35] that (cryptically) yielded Theorem 3.3(A)(C) and Theorem 3.7 were made in the terminology of "coloring graphs". In giving those arguments of Kászonyi here (in the proofs of Theorems 3.3 and 3.7), we have simply transcribed Kászonyi's own presentation of those arguments, involving the terminology of "coloring graphs", into the more transparent terminology of edge-3-colorings and Kempe cycles.

(E) To summarize, the work of Kászonyi [Ká1, Ká2, Ká3], along with some of the related later work of other people as described above, provide a collection of mathematical problems (including, but not limited to, the ones listed above) which can be attacked without too much specialized mathematical preparation, and which are in particular well suited for independent research projects for undergraduate mathematics students. This survey paper can hopefully facilitate research on such problems.

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